GINZBURG-LANDAU THEORY ON RIEMANN SURFACES

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Consider the Ginzburg-Landau equations functions on a bounded domain $\Omega \subset \mathbb{R}^2$:

$$\text{(GL)} \qquad \begin{cases} -\Delta_A \Psi = \kappa^2(1 - |\Psi|^2)\Psi, \\ d^* dA = \Im(\bar{\Psi}\nabla_A \Psi). \end{cases}$$

Here $\kappa > 0$ is a material constant, $\Psi(x)$ is a complex-valued function, A is a real-valued 1-form, $\nabla_A := \nabla + iA$ is the the covariant derivative induced by A, and d is the exterior derivative mapping p-forms to $p+1$ forms.

(GL) has $U(1)$-gauge symmetry, in the sense that if $\rho \in C^1(\Omega, U(1))$, then

$$T_\rho^{\text{gauge}} : (\Psi, A) \mapsto (\rho(x)\Psi(x),\ A - \rho^{-1}(x) d\rho(x))$$

maps solutions to solutions.

There are three associated physical quantities:

$$(1) \qquad \begin{cases} |\Psi|^2 \text{ is the local density of (Cooper pairs of) superconducting electrons,} \\ dA \text{ is the magnetic field,} \\ \Im(\bar{\Psi}\nabla_A\Psi) \text{ is the supercurrent density.} \end{cases}$$

The basic non-trivial solutions are called *magnetic vortices*. These are some local structure with finite energy and non-trivial topological degree. Call a solution (Ψ, A) to (GL) an *Abrikosov lattice* if the associated quantities in (1) are

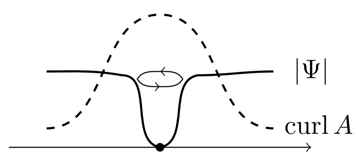

FIGURE 1. Here shows a cross section of a vortex solution Ψ, A near a core at $r = 0$, where the superconducting electron density $|\Psi|$ vanishes and the magnetic field curl A penetrates. For an N-vortex, the order parameter Ψ winds around the center N times, and the penetrating field has N quanta of magnetic flux.

all periodic w.r.t. some planar lattice $\lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$ (i.e. invariant under translation by elements in λ). In my last talk I showed (Ψ, A) is a lattice solution $\iff$

$$(2) \qquad \begin{cases} \Psi(x+s) = e^{ig_s(x)}\Psi(x), \\ A(x+s) = A(x) + \nabla g_s(x). \end{cases}$$

where g satisfies the cocycle condition:

$$(3) \qquad g_{s+t}(x) - g_t(x+s) - g_s(x) \in 2\pi\mathbb{Z}. \qquad (s, t \in \lambda).$$

The function $e^{ig_s(x)}$ is called *automorphy factor*. Two automorphy factors $e^{ig_s(x)}$ and $e^{ig'_s(x)}$ are said to be equivalent if they satisfy $g'_s(x) = g_s(x) + \chi(x+s) - \chi(x)$ for some function χ. A function Ψ that satisfies $T_s^{\text{trans}}\Psi = e^{g_s(x)}\Psi$ is said to be a $e^{ig_s(x)}$*-theta function*. Gunning in his classification of automorphy factors [3, Theorem 2] shows that every gauge-exponent g_s satisfying (2)-(3), is equivalent to

$$(4) \qquad \begin{cases} \dfrac{b}{2} s \cdot Jx + c_s, \quad (J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, b = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega dA) \\ c_{s+t} - c_s - c_t - \dfrac{b}{2} s \cdot Jx \in 2\pi\mathbb{Z}. \end{cases}$$

Assume well-posedness for the moment.

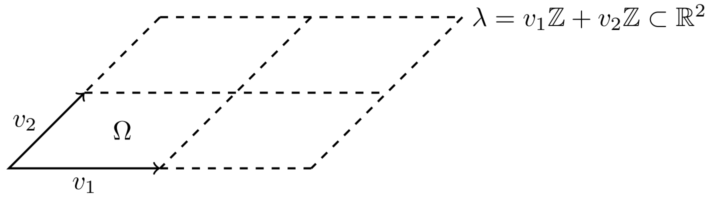

FIGURE 2. A fundamental cell Ω tessallates $\mathbb{R}^2$ under translations in the lattice group λ.

Toric geometry. Fix a fundamental cell Ω of the underlying lattice λ, and identify the opposite sides of the parallelogram Ω. Consider a vortex solution to (GL) on Ω. Then (2) allows one to extend this solution to the entire plane, since Ω tessellates $\mathbb{R}^2$.

Thus Abrikosov lattices can be viewed as vortices defined on the flat torus $\mathbb{T} = \mathbb{R}^2/\lambda$, which is homeomorphic to the standard torus through

$$\Omega \ni av_1 + bv_2 \mapsto (e^{2\pi ai}, e^{2\pi bi}).$$

Main result. In what follows, we show Abrikosov lattices satisfying (2) correspond to sections of and connection on $L \to \mathbb{T}$, where $L = (\mathbb{R}^2 \times \mathbb{C})/\lambda$ is the line bundle over complex torus. Here the action of λ is

$$(x, \Psi) \mapsto (x + s, e^{ig_s(x)}\Psi) \quad (s \in \lambda).$$

Note $L, \mathbb{T}$ are manifold which locally look like $\mathbb{R}^2 \times \mathbb{C}, \mathbb{R}^2$ resp.. L is non-trivial in the sense that $L \neq \mathbb{R}^2 \times V$ for any vector space V. Recall for a line bundle $L \xrightarrow{p} X$, a *section* is a map $s : X \to L$ s.th. $p \circ s = 1$. A *connection* ∇ maps sections on L to 1-forms on L, and satisfies Leibnitz rule $\nabla(fs) = f\nabla s + df \otimes s$.

Claim: there exists an 1-1 correspondence between equivariant states satisfying (2) and sections of and connections on L, given by

$$\phi([x]) = [(x, \Psi(x))], \quad \nabla\phi([x]) = [\nabla_A \Psi(x)], \tag{5}$$

where $\nabla_A \psi \sim \nabla_{A'}\Psi'$ if $(\Psi', A') = T^{\text{gauge}}_\rho(\Psi, A)$ for some ρ.

Proof. First check (5) is well-defined. If $x' = x + s$ for some $s \in \lambda$, then by (2) $\Psi' = \Psi(x + s) = e^{ig_s(x)}\Psi(x) \sim \Psi(x)$. Thus $(x', \Psi') \sim (x, \Psi)$. Similarly, $(\nabla_A\Psi)(x + s) = \nabla_{(A+\nabla g_s)} e^{ig_s(x)}\Psi(x) \sim \nabla_A\Psi$ through $T^{\text{gauge}}_{g_s}$.

It follows from the definition that (5) is 1-1. Conversely, given a section ϕ on L, construct an equivariant solution as follows. For $x \in \Omega$, since there is only one Ψ satisfying $\phi([x]) = [(x, \Psi)]$. Define $\Psi(x) = \Psi$. Then extend to $\mathbb{R}^2$ by (2) and some gauge exponent, say (3), which satisfies the cocycle condition (3). Similarly one can define 1-form A from a connection ∇ on L. □

Hyperbolic geometry. In a more genereal setting, one can consider (GL) on generic compact connected orientable Riemann surfaces, classified by genus g. We have discussed the cases for $g = 0$ (planar domain) and $g = 1$ (torus).

Let $\mathbb{H} := \{z \in \mathbb{C} | \Im z > 0\}$ be the Poincare half-plane, equipped with metric $ds = |dz|/\Im z$. This has Gaussian curvature -1 (hyperbolic). The group $SL(2, \mathbb{R})$ represented by Mobius transforms acts on $(\mathbb{H}, ds)$ as isometries. A *Fuchsian* group is a discrete subgroup of $PSL(2, \mathbb{R}) := SL(2, \mathbb{R})/\{\pm\mathbf{1}\}$. (E.g. $PSL(2, Z)$, the modular group.) One can show that if a compact RS has $g > 1$, then it is homeomorphic to $\mathbb{H}/\Gamma$ for some Fuchsian group Γ acting freely (i.e. no fixed point).

Let $L \xrightarrow{p} X$ be the line bundle $L := (\mathbb{H} \times \mathbb{C})/\Gamma$, where the action is

$$(x, \Psi) \mapsto (\gamma s, e^{ig_\gamma(x)}\Psi) \quad (\gamma \in \Gamma),$$

for some automorphy factor $g_\gamma(x)$ satisfying the cocycle condition

$$g_{\gamma\gamma'}(x) - g'_\gamma(\gamma x) - g_\gamma(x) \in 2\pi\mathbb{Z}. \qquad (\gamma, \gamma' \in \Gamma). \tag{6}$$

To generalize the notion of Abrikosov lattice, call (Ψ, A) an Γ-equivariant solution iff

$$\begin{cases} \Psi(\gamma x) = e^{ig_\gamma(x)}\Psi(x), \\ A(\gamma x) = A(x) + dg_\gamma(x), \end{cases} \tag{7}$$

for some automorphy factor g_γ satisfying (6). The problem now is how to calculate the automorphy factor g_γ in terms of the connection A, in a fashion similar to (2). This is done for instance in [2]. See also lecture notes [4, Section 14].

Basic existence result [1]. $\kappa = 1/\sqrt{2}$, $|\Omega| > 4\pi N \implies$ there exists solution (Ψ, A) to (GL) s.th. $\deg \Psi = \frac{1}{|\Omega|}\int_\Omega dA = N$. These solutions are called N vortices.

For $\kappa = 1/\sqrt{2}$, using intergration by parts one can show that the energy functional split into two parts:

$$\begin{aligned} E(A,\Psi) &:= \frac{1}{2}\int_\Omega \left\{ |\nabla_A \Psi|^2 + (\operatorname{curl} A)^2 + \frac{1}{4}(|\Psi|^2-1)^2 \right\} \\ &= \frac{1}{2}\int_\Omega \Big\{ ((\partial_1\Psi_1 + A_1\Psi_2) - (\partial_2\Psi_2 - A_2\Psi_1))^2 + ((\partial_2\Psi_1 + A_2\Psi_2) - (\partial_1\Psi_2 - A_1\Psi_1))^2 + \\ &+ (\operatorname{curl} A + \frac{1}{2}(\Psi_1^2 + \Psi_2^2 - 1))^2 \Big\} + \frac{1}{2}\int_\Omega \operatorname{curl} A \\ &\geq \frac{1}{2}\int_\Omega \operatorname{curl} A = \pi N. \end{aligned}$$

The first part is a sum of squares, and the second part gives a lower bound on the energy by the topological quantity N. This equality is attained iff the first integral is zero, i.e.,

$$\begin{aligned} (\partial_1\Psi_1 + A_1\Psi_2) - (\partial_2\Psi_2 - A_2\Psi_1) &= 0. \\ (\partial_2\Psi_1 + A_2\Psi_2) - (\partial_1\Psi_2 - A_1\Psi_1) &= 0, \\ \operatorname{curl} A + \frac{1}{2}(\Psi_1^2 + \Psi_2^2 - 1) &= 0. \end{aligned} \tag{8}$$

These are called the *Bogomolny equations*.

Consider the third Bogolmony equation

$$\operatorname{curl} A + \frac{1}{2}(|\Psi|^2 - 1) = 0 \iff \operatorname{curl} A = \frac{1}{2}(1 - |\Psi|^2).$$

Integrating this over Ω, one gets an upperbound on the vortex number N in terms of the area of the domain:

$$2\pi N = \int_\Omega \operatorname{curl} A = \int_\Omega \frac{1}{2}(1-|\Psi|^2) < \int_\Omega \frac{1}{2} = \frac{1}{2}|\Omega| \iff |\Omega| > 4\pi N. \tag{9}$$

This is called the *Bradlow condition.* In [1], Bradlow shows that the upperbound in (9) holds if Ω is replaced by a compact Kähler manifold of arbitrary dimension. (To derive this in the general setting, the Bogolmony equation has to be modified appropriately.)

References

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