Mathematical concepts of the Ginzburg-Landau theory of superconductivity

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February 29, 2020

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Here $\kappa > 0$ is a material constant, $\Psi(x)$ the *order parameter*, $A(x)$ the *vector potential*, and $\nabla_A := \nabla - iA$ the covariant derivative that couples A to Ψ.

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assuming the material has perfect response, energy (1) becomes

$$E_\Omega^h(\Psi, A) := \frac{1}{2}\int_\Omega \left\{ |\nabla_A\Psi|^2 + |\operatorname{curl} A - H|^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}. \tag{2}$$

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One is interested in the phase transition as external field strength

$h \geq 0$ is lowered.

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The normal, non-superconducting state is given by

$$\Psi_0 \equiv 0, \qquad \operatorname{curl} A_0 \equiv H,$$

where H is the constant applied field.

The GL free energy for the normal and the purely superconducting states are respectively given by

$$E_0(\kappa, \Omega) = \frac{\kappa^2}{4}|\Omega|, \qquad E_s(h, \Omega) = \frac{h^2}{2}|\Omega|. \tag{3}$$

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Spectral analysis on the linearized problem suggests an inhomogeneous solution can bifurcate from the normal solution at $h = h_{c_2} := \kappa^2$, called the *upper critical field*.

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On the other hand, for $\kappa > 1/\sqrt{2}$, there is an interval
$h_{c_1} < h < h_{c_2}$ in which an interfacial state is possible. Thus one
expects to observe a gradual, *second-order phase transition*. In this
case we say the superconducting material is of *type II*.

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$$\Psi = \Psi_N = f_N(r)e^{iN\theta}, \quad A = A_N = a_N(r)\nabla(N\theta), \tag{4}$$

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where (r, θ) is the polar coordinate of $x \in \mathbb{R}^2$, $f_N(0) = 0$, and N is an integer. These solutions are called *N-vortices*, and N the *vortex number*. For superconductors, a vortex solution describes the mixed state, with N quanta of magnetic flux and the normal phase residing where the vortex vanishes.

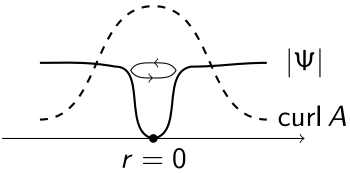

Figure above shows a cross section of a vortex solution near a core at $r = 0$, where the superconducting electron density $|\Psi|$ vanishes and the magnetic field $\operatorname{curl} A$ penetrates.

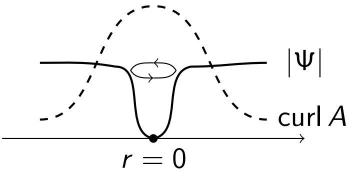

Figure above shows a cross section of a vortex solution near a core at $r = 0$, where the superconducting electron density $|\Psi|$ vanishes and the magnetic field $\operatorname{curl} A$ penetrates. For an N vortex, the order parameter Ψ winds around the center N times, and the penetrating field has N quanta of magnetic flux.

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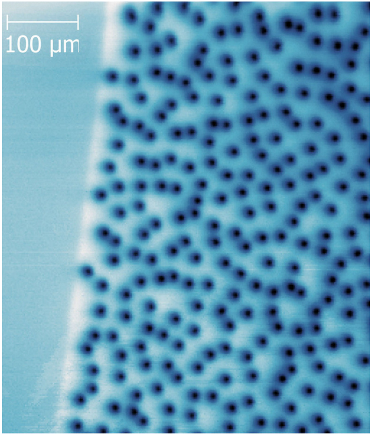
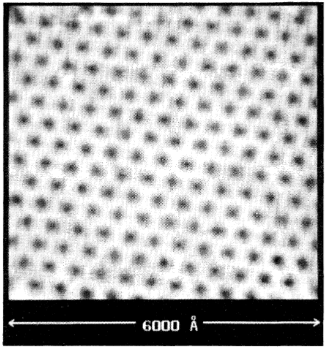


(a) Vortices are seen as round dark spots. F. S. Wells et al., Sci. Rep. 2015; 5: 8677 (2015).



(b) Vortices (in black) forming an Abrikosov lattice. H. F. Hess et al., Phys. Rev. Lett. 62, 214 (1989).

Mathematical aspects

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Equations (GL) have three classes of important symmetries:

$$T_s^{\text{trans}} : (\Psi(x), A(x)) \mapsto (\Psi(x+s), A(x+s)) \quad (s \in \mathbb{R}^d),$$

$$T_R^{\text{rot}} : (\Psi(x), A(x)) \mapsto (\Psi(R^{-1}s), RA(R^{-1}s)) \quad (R \in O(d)),$$

$$T_g^{\text{gauge}} : (\Psi(x), A(x)) \mapsto (e^{ig(x)}\Psi(x), A(x) + \nabla g(x)) \quad (g \in C^1(\mathbb{R}^d)).$$

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References

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