ABRIKOSOV LATTICE SOLUTIONS TO THE GINZBURG-LANDAU EQUATIONS

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Abstract. Ginzburg-Landau theory describes a superconductor below critical temperature at the macroscopic level. The fundamental solution to the Euler-Lagrange equations (a pair of nonlinear PDEs known as Ginzburg- Landau equations) are called vortices. These are localized structures with nontrivial topological degrees. In 1957 Abrikosov discovered an important class of solutions in which vortices arrange themselves as a lattice, called Abrikosov lattices. From here Abrikosov predicted the mixed state of the type II superconductors, which were later verified by experiments. In this talk we review the mathematical concepts of GL theory and Abrikosov lattices.

1. The Ginzburg-Landau theory of superconductivity

The Ginzburg-Landau equations. As far as the mathematics is concerned, it is customary to consider the idealized situation of a space-filling superconductor that is homogeneous along one direction. In this setting, we look at the plane normal to the direction where the magnetic field is approximately constant, and the two-dimensional *Ginzburg-Landau equations* are given in terms of a pair of functions $(\Psi, A) : \Omega \subset \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2$ as:

$$\begin{cases} -\Delta_A \Psi = \kappa^2(1 - |\Psi|^2)\Psi, \\ \operatorname{curl}^* \operatorname{curl} A = \Im(\bar{\Psi}\nabla_A \Psi). \end{cases} \tag{GL}$$

Here $\kappa > 0$ is a material constant, and Ω an open, connected domain. $\Psi(x)$ is called the *order parameter*, $A(x)$ the *vector potential*, and

$\nabla_A := \nabla - iA$ is the covariant derivative,
$\nabla_A^* = -\operatorname{div} + iA$ is its adjoint,
$-\Delta_A := \nabla_A^* \nabla_A$.
For a vector field $A : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\operatorname{curl} A := \partial_1 A_2 - \partial_2 A_1$ is the two-dimensional curl.
For a scalar field $B : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, $\operatorname{curl}^* B = (\partial_2 B, -\partial_1 B)$ is the adjoint of curl.

(GL) are the Euler-Lagrange equations of the *Ginzbug-Landau free energy*, which describes the difference in Helmholtz free energy between normal and superconducting states:

$$E_\Omega(\Psi, A) := \frac{1}{2}\int_\Omega \left\{ |\nabla_A \Psi|^2 + (\operatorname{curl} A)^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}. \tag{1}$$

This theory, proposed by Ginzburg and Landau in 1950 [9], is built on Landau's theory of phase transition and captures the desired phenomenology, as we will see below. There are three associated physical quantities in the energy functional:

$$\begin{cases} |\Psi|^2 \text{ is the local density of (Cooper pairs of) superconducting electrons,} \\ \operatorname{curl} A \text{ is the magnetic field,} \\ \Im(\bar{\Psi}\nabla_A \Psi) \text{ is the supercurrent density.} \end{cases} \tag{2}$$

Thus the second equation in (GL) is Ampere's law. In the presence of an external magnetic field (homogeneous along the same direction as $\operatorname{curl} A$), we assume the material has perfect response. Then the free energy (1) is modified as

$$E_\Omega^h(\Psi, A) := \frac{1}{2}\int_\Omega \left\{ |\nabla_A \Psi|^2 + (\operatorname{curl} A - h)^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}, \tag{3}$$

where $h \geq 0$ is the external field strength. Using Sobolev inequalities, one can show that if $\Omega \subset \mathbb{R}^d$ is bounded (for instance, as the section of a finite superconductor or a fundamental cell of a lattice solution), then for all h, (3) defines a $\mathbb{R}$-valued, C^1 functional on the space $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ [1, Section 1.2], whose Euler-Lagrange equations are (GL).

Symmetry. Equations (GL) have three classes of important symmetries. Let (Ψ, A) be a solution to (GL). Under translation

$$T_s^{\text{trans}} : (\Psi(x), A(x)) \mapsto (\Psi(x+s), A(x+s)) \quad (s \in \mathbb{R}^2),$$

rotation

$$T_R^{\rm rot} : (\Psi(x), A(x)) \mapsto (\Psi(R^{-1}s), RA(R^{-1}s)) \quad (R \in O(2)),$$

and gauge transform

$$T_g^{\rm gauge} : (\Psi(x), A(x)) \mapsto (e^{ig(x)}\Psi(x), A(x) + \nabla g(x)) \quad (g \in H^2_{\rm loc}(\mathbb{R}^2, \mathbb{R})),$$

the resulting functions are still solutions to (GL). In contrast to the global (ungauged) theory, for which constant energy solutions all have constant phase at infinity, the $U(1)$-gauge symmetry of GL theory allows the existence of finite energy solution with nontrivial topological degree [19, Section 7.2]. These nontrivial solutions are the main interest in the remaining of this paper.

Trivial solutions of (GL). There are two trivial solutions of (GL). The first one is

$$\Psi_s \equiv 1, \qquad A_s \equiv 0,$$

which corresponds to the purely superconducting state. This solution reflects the Meissner effect that characterizes superconductivity: the expulsion of the external magnetic field from the bulk of the superconducting material. The second one is

$$\Psi_0 \equiv 0, \qquad \operatorname{curl} A_0 \equiv h,$$

which corresponds to the normal, non-superconducting state. Note here we assume the material responds perfectly to the applied field, and the constant $h = \operatorname{curl} A$ is determined by the external field strength. One possible choice of the normal state vector potential is $A_0 = \frac{h}{2}(-x_2, x_1)$.

One is interested in when a nontrivial solution can bifurcate from the trivial ones. For some heuristic insight, we linearize (GL) at the normal state (A_0, Ψ_0). Rescale $\tilde{A}_0 = A_0/h$ and write $\tilde{A} = \tilde{A}_0 + \epsilon\alpha$, $\tilde{\Psi} = \sqrt{\epsilon}\Psi$. Taking the first order terms as $\epsilon \to 0$, we have the linearized (GL) as:

$$\begin{cases} (-\Delta_{h\tilde{A}_0} - \kappa^2)\tilde{\Psi} = 0, \\ \operatorname{curl}^* \operatorname{curl} \alpha = \Im(\bar{\tilde{\Psi}} \nabla_{h\tilde{A}_0} \tilde{\Psi}). \end{cases} \tag{4}$$

The first equation is the eigenvalue problem for a Schrödinger operator, and the solution is well known: the bottom of the spectrum of $-\Delta_{h\tilde{A}_0}$ is h, and therefore $-\Delta_{h\tilde{A}_0} - \kappa^2$ has zero eigenvalue if $h = h_{c_2} := \kappa^2$. The value h_{c_2} is called the *upper critical field*, and above computation suggests a nontrivial solution can bifurcate from the normal solution at h_{c_2}. This leads to the existential problem for the *mixed states*. Some rigorous results are given in Section 4.

Type I and type II superconductors. Consider the energy functional (3) in the presence of external magnetic field. The free energy for the purely superconducting and the normal states are respectively given by

$$E_s = \frac{\kappa^2}{4}|\Omega|, \qquad E_0 = \frac{h^2}{2}|\Omega|. \tag{5}$$

For $h > h_{c_1} := \kappa/\sqrt{2}$, the normal state is the global minimizer of (3). The value h_{c_1} is called the *lower critical field*. For $\kappa < 1/\sqrt{2}$, we see that $h_{c_2} < h_{c_1}$. Thus as the previous computation shows, when the mixed state can bifurcate from the normal state, the purely superconducting state is already energetically favourable. In this case one expects to observe an abrupt, *first-order phase transition* when h is lowered, and we say the superconducting material is of *type I*. (Pure metals belong to this category.) On the other hand, for $\kappa > 1/\sqrt{2}$, there is an interval $h_{c_1} < h < h_{c_2}$ in which an interfacial state is possible. Thus one expects to observe a gradual, *second-order phase transition*. In this case we say the superconducting material is of *type II*. (Alloys and dirty metals belong to this category). Of both physical and mathematical interest is the critical threshold $\kappa = 1/\sqrt{2}$, called the *self-dual* case, for which there exist a rich family of solutions to (GL).

Length scales and the parameter κ. The material parameter κ in (GL) comes from the length scales of the problem. We give a heuristic derivation following Gustafson [15]. Let (Ψ, A) be a finite energy solution to (GL). Then the following boundary conditions at infinity must be satisfied:

$$\begin{cases} |\Psi| \to 1, \\ \nabla_A \Psi \to 0, \end{cases} \qquad (|x| \to \infty). \tag{6}$$

Write Ψ in the polar form $\Psi = |\Psi|e^{i\phi}$. Put $g := 1 - |\Psi|$, $B := \operatorname{curl} A$. Taking curl of both sides of the Ampere's equation in (GL), we have

$$(-\Delta + (1-g)^2)B = -2(1-g)\operatorname{curl} g \cdot (\nabla\phi - A). \tag{7}$$

Since

$$|\nabla_A \Psi|^2 = |\nabla g|^2 + (1-g)^2(\nabla\phi - A|^2,$$

and since $1-g=|\Psi|\to 1$ as $|x|\to\infty$ from (6), it follows $\nabla\phi - A\to 0$ as $|x|\to\infty$. Since $g=1-|\Psi|\to 0$, at the leading order in g, (7) becomes

$$(-\Delta+1)B=-2\operatorname{curl} g\cdot(\nabla A-\phi)\quad(|x|\to\infty).$$

Now $\operatorname{curl}(\nabla\phi-A)=-\operatorname{curl}A=-B$, so $\nabla A-\phi$ decays at the same rate as B and the r.h.e. of (7) decays at least as fast as B (since $\operatorname{curl} g\to 0$ at infinity). Thus $B\sim e^{-\alpha|x|}$ where $\alpha=1$, as $|x|\to\infty$, because the Green function of $-\Delta+\kappa^2$ decays as $e^{-\kappa r}$ for large $r:=|x-y|$.

Next, at the leading order in g, the first equation in (GL) becomes

$$(-\Delta+\kappa^2)g=-|\nabla\phi-A|^2.$$

We have derived that r.h.e. is of the order $e^{-2|x|}$. Considering the decay rate of the Green function, we see $g\sim e^{-\beta|x|}$ where $\beta:=\min(\kappa,2)$, as $|x|\to\infty$.

Physically, the reciprocal $\eta:=1/\alpha$ measures the scale at which the magnetic field B varies, called the *penetration depth*. The reciprocal $\xi:=1/\beta$ measures the scale at which the modulus of the order parameter $|\Psi|$ varies, called the *coherence length*. The material parameter in (GL) is proportional to the ratio of penetration depth and coherence length, $\kappa:=\frac{\eta}{\sqrt{2}\xi}$. At the self-dual case $\kappa=1/\sqrt{2}$, one sees that the two length scales η, ξ coincide.

In particular, the N-vortices in (8) demonstrate exponential localization. In Jaffe and Taubes [16, Sectionss. III.6-7] the following asymptotics are established for N-vortices :

$$\begin{aligned}&|1-f_N(r)|\le Ce^{-\alpha r},\\ &\operatorname{curl}A_N(r)=N\beta K_1(r)\left(1-\frac{1}{2}r^{-1}+O(r^{-2})\right),\end{aligned}$$

where $C=C(A,\Psi)$, $\alpha=\alpha(\kappa)$, $\beta=\beta(N)$ are some constants and $K_1(r)$ is the modified Bessel function of the second type. This fast localization serves as the basis for constructing approximate N-vortex solutions to (GL) by patching together N simple vortices. See Gustafson and Sigal [14] for its application to derive effective dynamics.

The existential theory of vortices. In 1958, Ginzburg and Pitaevskii conjectured in [8] that there exists solutions with radial symmetry to (GL) of the form

$$\Psi=\Psi_N=f_N(r)e^{iN\theta},\quad A=A_N=a_N(r)\nabla(N\theta),\tag{8}$$

where (r,θ) is the polar coordinate of $x\in\mathbb{R}^2$, $f_N(0)=0$, and N is an integer. (Since (GL) is invariant under translation, we set the origin to be the center of the profile functions.) These solutions are called *N-vortices*, and N the *vortex number*. For superconductors, a vortex solution describes the mixed state of the material, with N quanta of magnetic flux and the normal phase residing where the vortex vanishes. This conjecture was proved subsequently, and the prediction can be validated in experiments.

L. Onsager conjectured that as these solitons describe vortices in superfluids, the magnetic flux

$$N:=\frac{1}{2\pi}\int_\Omega\operatorname{curl}A\tag{9}$$

should be quantized, in contrast to vortices in normal fluids ([21, p.191]). Using homotopy theory, one can show that for solutions of the form (8) with sufficiently regular asymptotics, the magnetic flux is indeed an integer.

In Section 3, we prove an analogous result for lattices as a consequence of the cocycle condition (20).

Note that the normal state (Ψ_0,A_0) has $N=0$. Since $(\Psi(x),A(x))\to(\Psi(-x),-A(-x))$ is a symmetry of (GL), w.l.o.g. one can assume $N\ge 0$.

The Bogomolny regime. In the self-dual case $\kappa=1/\sqrt{2}$, vortices can localize at any given points in the plane effecively without interaction and remain static. Identify $\mathbb{R}^2\cong\mathbb{C}$ so that Ψ, A are defined on the complex plane. The main result concerning the existence of the static multi-vortex solutions is the following:

Theorem 1.1. *Let $\kappa=1/\sqrt{2}$, $z_1,\dots,z_N\in\mathbb{C}$ (counting multiplicity). Then* (GL) *has a unique (up to gauge symmetry) solution s.th.*

(1) the solution is C^∞;
(2) the zero set $Z(\Psi)=\{z_1,\dots,z_N\}$;
(3) the vortex number $N\in\mathbb{N}$.

Remark. We understand a vortex localizes at each zero z_n, $n=1,\dots,N$. Later we show that a multi-vortex solution is characterized by this zero set. This configuration is stable in approriate sense under both first- and second-order dynamics, as we shall explain later.

The essential feature of the self-dual case is that a sequence of reductions is possible at $\kappa=1/\sqrt{2}$. To begin with, write $\Psi=\Psi_1+i\Psi_2$ in terms of its real and imaginary parts. Bogomolny discovered in [2] that for $\kappa=1/\sqrt{2}$, using intergration by parts one has the following lower bound on the energy functioal:

$$E(A,\Psi) = \frac{1}{2}\int \Big\{((\partial_1\Psi_1 + A_1\Psi_2) - (\partial_2\Psi_2 - A_2\Psi_1))^2 + ((\partial_2\Psi_1 + A_2\Psi_2) - (\partial_1\Psi_2 - A_1\Psi_1))^2 +$$
$$+(\operatorname{curl} A + \frac{1}{2}(\Psi_1^2 + \Psi_2^2 - 1))^2\Big\} + \frac{1}{2}\int \operatorname{curl} A$$
$$\geq \frac{1}{2}\int_{\mathbb{R}^2} \operatorname{curl} A = \pi N.$$

The equality is attained iff the first integral is zero, i.e.

$$(\partial_1\Psi_1 + A_1\Psi_2) - (\partial_2\Psi_2 - A_2\Psi_1) = 0.$$
$$(\partial_2\Psi_1 + A_2\Psi_2) - (\partial_1\Psi_2 - A_1\Psi_1) = 0, \tag{10}$$
$$\operatorname{curl} A + \frac{1}{2}(\Psi_1^2 + \Psi_2^2 - 1) = 0.$$

Next, omplexify as

$$z = x_1 + ix_2, \quad \partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar\partial = \frac{1}{2}(\partial_1 + i\partial_2), \quad A = (\alpha + \bar\alpha, i(\alpha - \bar\alpha)),$$

where $\alpha = (A_1 - iA_2)/2$, $\bar\alpha = (A_1 + iA_2)/2$. Then

$$\nabla_A = ((\partial - i\alpha) + (\bar\partial - i\bar\alpha), i(\partial - i\alpha) - i(\bar\partial - i\bar\alpha)),$$

and the first two equations in (10) become the real and imaginary parts of a single equation,

$$2(\bar\partial - i\bar\alpha)\Psi = 0. \tag{11}$$

From (11) one can solve A in terms of Ψ. Indeed, for $\Psi \neq 0$,

$$\alpha = i\partial \ln \bar\Psi. \tag{12}$$

Suppose α is continuous. Then one can extend α to the zero set of Ψ, $Z(\Psi)$. Later we will show this ansatz is satisfied. In fact, using a standard bootstrap argument for elliptic regularity, one can show α is C^∞.

Witten suggested in [25] that one can reformulate (11) in terms of $\ln|\Psi|^2$, whereby one arrives at a highly nonlinear elliptic equation. Moreover, $\ln|\Psi|^2$ is singlular precisely where Ψ has a zero. Indeed, for fixed $z_1,\dots,z_N \in \mathbb{C}$, put

$$\Psi =: e^{(u+i\Theta)/2}, \quad \Theta = 2\sum_{n=1}^{N} \arg(z - z_n). \tag{13}$$

(Note here Θ is multivalued up to multiples of 2π.) Then the third equation in (10) becomes an inhomogeneous Liouville equation in u with zero boundary condition at infinity:

$$\begin{cases} -\Delta u + e^u - 1 = -4\pi\sum \delta(z - z_n) \quad \text{(Dirac delta)}, \\ \lim_{|z|\to\infty} u(z) = 0. \end{cases} \tag{14}$$

(Here and below summations and products are taken over $n = 1,\dots,N$.) Introduce a parameter $\mu > 0$ and put

$$u_0 = u_0^\mu := -\sum \ln(1 + \mu|z - z_n|^{-2}).$$

Then the distributional derivatives of u_0 satisfy

$$-\Delta u_0 = \underbrace{4\sum \mu(|z - z_n|^2 - \mu)^{-2}}_{\text{call this } g_0 = g_0^\mu} - 4\pi\sum \delta(z - z_n).$$

Upon substituting $v := u - u_0$, the BVP above becomes

$$\begin{cases} \Delta v = e^{v+u_0} + (g_0 - 1) \quad \text{(Dirac delta)}, \\ \lim_{|z|\to\infty} v(z) = 0. \end{cases} \tag{15}$$

By construction, u_0 is C^∞ except for at $z_1,\dots,z_N$. Therefore if $v \in C^\infty$ solves (15), then u and u_0 must have the same set of poles, which also coincides with $Z(\Psi)$ through (13).

The moduli space. The natural problem to ask after one solves a variational problem is to charaterize the minimizers. In our case, this amounts to finding the *moduli space* of (GL), the solution space of multi-vortices mudulo the action of gauge symmetry. One can show that the moduli space of N-vortices is the N-fold symmetric product $\mathbb{C}^N/S_N$ But through the formula

$$p(z) = \prod(z - z_n) = z^N + \sum_{n=1}^{N} c_i z^{N-i},$$

where c_i are the i-th elementary symmetry polynomials in $z_1, \ldots, z_N$, one can relabel a solution with the *ordered* pairs $(c_1, \ldots, c_N) \in \mathbb{C}^N$. This proves the claim that the moduli space of N-vortices is $\mathbb{C}^N$.

One implication is that the vortex number N defined by (9) equals to the *topological degree* of Ψ,

$$\deg \Psi = \text{winding number of } \left.\frac{\Psi}{|\Psi|}\right|_{|x|=R}$$

which is well defined for $R \gg 1$. To see this, we show $Z(\Psi) = S(\Theta)$, the singularities of Θ given by (13). Then one can read off the residual from the formula for Θ. One inclusion is clear: for Ψ to be C^∞, it must be that the singularities of Θ occur at points where Ψ vanishes. Thus $Z(\Psi) \supset S(\Theta)$. On the other hand, near each $z_n \in \mathbb{Z}(\Psi)$,

$$\Psi(z) = e^{(u+i\Theta)/2} = |z - z_n|^{p_n} |h(z)| e^{i(p_n \arg(z - z_n) + \arg h_n}.$$

Since $h_n \in C^\infty$, so is $\arg h_n$ modulo 2π. Thus Θ is singular at z_n, and $Z(\Psi) \subset S(\Theta)$. The claim follows.

Another implication of the moduli space is the right-angle scattering. Physically, one would expect to observe that when N vortices collide, they scatter at the angle of π/N. In the moduli space $\mathbb{C}^N$, after possibly reparametrization, a smooth trajectory is given by

$$(0, \ldots, 0, t) \qquad (t \geq 0).$$

The corresponding polynomial is

$$p(z) = z^N + t,$$

whose roots are $z_n = |t|^{1/N} e^{2\pi i n/N}$, $n = 1, \ldots, N-1$. This implies that after collision (i.e. $p(z) = 0$)), the outgoing trajectory is rotated by π/N w.r. to the incoming trajectory. Much more can be said about the dynamics in the moduli space, some of which we allude to below.

Dynamics of vortices. The simplest kind of dynamics in Ginzburg-Landau theoy is given by the (L^2-)gradient flow of the energy functional (1),

$$\partial_t u = -E'(u), \quad u = (\Psi, A).$$

This gives the Gorkov-Eliashberg equations [10]:

$$\text{(GE)} \qquad \begin{cases} \partial_t \Psi = \Delta_A \Psi + \kappa^2 (1 - |\Psi|^2) \Psi, \\ \partial_t A = -\operatorname{curl}^* \operatorname{curl} A + \Im(\bar{\psi} \nabla_A \Psi). \end{cases}$$

The wellposedness of the gradient flow dynamics is established by Demoulini and Stuart in [5]. In particle physics, to obtain an Lorentz invariant theory, one would use the second-order dynamics. The corresponding equations are known as the *Maxwell-Higgs equations*, which, after choosing the temporal gauge, amounts to replacing the first-order time derivatives in (GE) with second-order ones. In addition to this, Manton has suggested in [18] another dynamical model, called the *Chern-Simons-Schrodinger equations*, that takes into consideration the complex structure of the multi-vortex configuration space, in which Ψ evolves according to a nonlinear Schrodinger equation. This is studied by Demoulini and Stuart in [7, 24] and is applied to model condensed matter physics and quantum Hall effect at very low temperature.

There are two main directions for the dynamical problems. One way to go would be to derive a finite-dimensional *effective dynamics* for the infinite-dimensional evolution equations (GE) and its variants. In a setting similar to the GL theory, Manton suggested in [17] that such dimension reduction can be done by geodesics approximation on the moduli space. This idea was later formulated rigorously by D.M.A. Stuart in [6, 7]. Along this line of work we have the following results:

(1) in [6], Demoulini and Stuart show that for $\kappa = 1/\sqrt{2} + \epsilon$ with sufficiently small $\epsilon > 0$ and a solution to the Maxwell-Higgs equations with initial configuration given by an N-vortex, the vortex parameter of the solution to can be approximate by geodesics (in approrpriate sense) on the moduli space;
(2) later on, in [7], the same authors show an analogous result for the solution to the Chern-Simons-Schrodinger equations;
(3) in [14], Gustafson and Sigal compute the evolution equations for the vortex parameter (i.e. the points of localization) for $\kappa > 1/\sqrt{2}$ and solution to (GE) with initial configuration given by a widely-separated, approximate N-vortex state (obtained by patching N simple vortices together, since the localization is exponential).

Another direction would be to study various modes of stability of multi-vortex solutions under different dynamics. So far one can show that

(1) under first- and second-order dynamics, for $\kappa > 1/\sqrt{2}$, an N-vortex solution is orbitally stable (i.e. remains close for all time to the manifold generated by the symmetry group acting on the initial configuration) iff $|N| = 1$;
(2) under both dynamics, for $\kappa < 1/\sqrt{2}$, all vortex solutions are orbitally stable. This and the previous result are proved by Gustafson and Sigal in [12, 13];
(3) under (GE) dynamic, these results can be improved to asymptotic stability. This is proved by Gustafson and F. Ting in [13].

It remains open (as of [20]) to find the cretaria of asymptotic stability for the Maxwell-Higgs dynamic.

Interfacial tension. There is a phenomenological explanation for the above dynamical results. Consider the surface tension σ of the interface between normal and superconducting phases, which is defined by Ginzburg and Landau in [9] as the difference between the free energy (3) of the interfacial profile and either uniform phase at the lower critical magnetic field, $h_{c_1} = \kappa/\sqrt{2}$:

$$\sigma(\Psi, A) = E_\Omega^{h_{c_1}}(\Psi, A) - E_\Omega^{h_{c_1}}(0, a_{c_1}),$$

where $\operatorname{curl} a_{c_1} = h_{c_1}$ is the perfectly responding magnetic field. (Recall that the two uniform phases at h_{c_1} have the same energy, as in (5).) At the first place, the material is assumed to be a bulk superconductor and we are looking at the planar section normal to the direction, say z-axis, along which the material is approximately homogeneous. Now assume the solution 1. remains uniform along each section normal to the z-axis, 2. tending to the superconducting phase as $z \to \infty$ and 3. to the normal phase as $z \to -\infty$. The interface takes place at the plane $z = 0$. In symbols, we are looking for solutions that depend on the z-variable only, and satisfy the boundary conditions as follows:

$$(\psi(z), a(z)) = (\Psi(x, y; z), A(x, y; z)), \quad \lim_{z\to\infty}(|\psi|, \operatorname{curl} a) = (1, 0), \quad \lim_{z\to-\infty}(|\psi|, a) = (0, h_{c_1}). \tag{16}$$

The free energy (3) is then modified so as to integrate over the z-direction only, and the interfacial tension becomes

$$\sigma(\Psi, A) = \int_{\mathbb{R}} g(\psi, a) - g(0, a_c)\, dz, \quad g(\psi, a) = \frac{1}{2}\left(|\nabla_a \psi|^2 + (\operatorname{curl} a - h_{c_1})^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2\right).$$

Here the excess free energy is understood as per unit area in xy-plane.

Interfacial profiles that satisfy (16) are studied by S.J. Chapman in [3]. The author calculates that the interfacial tension $\sigma > 0$ if $\kappa < 1/\sqrt{2}$ (type I superconductors), $\sigma < 0$ if $\kappa > 1/\sqrt{2}$ (type II superconductors), and $\sigma = 0$ if $\kappa = 1/\sqrt{2}$ (self-dual case). One would therefore expect that drived by the interfacial tension, non-simple (vortex number $|N| > 1$) type II vortices would "destabilize" and expel from each other, and type I vortices would "regularize" and eventually coalesce. Whereas in the self-dual case, since there is no interfacial tension, vortices are static.

2. Abrikosov lattices

Let $(\Psi, A) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2$ be a solution to (GL). Given a planar lattice $\lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$ ($v_1, v_2 \in \mathbb{R}^2$), say (Ψ, A) is a λ-*lattice solution* if the associated quantities in (2) are all periodic w.r. to λ (i.e. invariant under translation by elements in λ.) In this case, λ is called the *underlying lattice* of the solution, and v_1, v_2 are called the *basis* of the lattice.

Our first result gives a working definition for (Ψ, A) to be a lattice solution.

Proposition 2.1. *(Ψ, A) is a λ-lattice solution iff for every $s \in \lambda$, there is $g_s \in H^2_{loc}(\mathbb{R}^2, \mathbb{R})$ s.th.*

$$\begin{cases} \Psi(x + s) = e^{ig_s(x)}\Psi(x), \\ A(x + s) = A(x) + \nabla g_s(x). \end{cases} \tag{17}$$

Proof. Suppose (Ψ, A) satisfy (17), then direct computation shows the three quantities in (2) are doubly periodic w.r. to λ.

Conversely, take $s \in \lambda$ and suppose magnetic field $\operatorname{curl} A$ is λ-periodic. Then the change from $A(x) \to A(x + s)$ must be a closed form, i.e. there is g'_s s.th. $A(x + s) = A(x) + \nabla g'_s(x)$. Write Ψ in the polar form $\Psi = |\Psi|e^{i\phi}$ and suppose density $|\Psi|^2$ and current $J := \Im(\bar{\Psi}\nabla_A\Psi) = |\Psi|^2(\nabla\phi - A)$ are λ-periodic. Then $\nabla\phi(x + s) = \nabla\phi(x) + \nabla g'_s(x)$ so as to cancel the change in A, and there is $g_s = g'_s + c_s$ (some constant) s.th. $\phi(x + s) = \phi(x) + g_s(x)$. □

In other words, (Ψ, A) is a λ-lattice solution iff it is gauge-equivariant (or "gauge-periodic") w.r. to λ. One can think of a lattice solution as a (multi-)vortex solution defined on a fundamental cell $\Omega \subset \mathbb{R}^2$ of the underlying lattice λ, and then extended to the entire plane by the gauge-periodicity in (17). This way one can avoid the problem that the Ginzburg-Landau free energy (1) of a lattice solution $E_{\mathbb{R}^2}(\Psi, A)$ over the entire plane is infinite, though $E_\Omega(\Psi, A)$ restricted to a fundamental cell (a bounded domain) is well-defined and smooth, as stated in Section 2.

Next, we state a rigorous existential result for lattices.

Theorem 2.2. *Let $\lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$ ($v_1, v_2 \in \mathbb{R}^2$) be a planar lattice, $\Omega \subset \mathbb{R}^2$ a fundamental cell of λ. Suppose*

$$|1 - b/\kappa^2| \ll 1, \quad (\kappa - \kappa_c(\lambda))(\kappa^2 - b) > 0, \tag{18}$$

where $b := \frac{1}{2\pi}\int_\Omega \operatorname{curl} A$ is the average magnetic flux per cell, and $\kappa_c(\lambda)$ is a certain critical threshold. Then $\kappa_c(\lambda) < 1/\sqrt{2}$, and there exists a C^∞ λ-lattice solution (Ψ, A) to (GL).

Remark. It is well known from experiment that type II superconductor ($\kappa > 1/\sqrt{2}$) can demonstrate mixed state described by Abrikosov lattice. However, condition (18) can also be satisfied with suitable underlying lattice for type I superconductor with material parameter $\kappa_c(\lambda) < \kappa < 1/\sqrt{2}$, provided the average flux per cell is close to the second critical magnetic field, $h_{c2} := \kappa^2$. (This is very high in reality, as used for Maglev.)

The proof of Theorem 2.2 is postponed to Section 5.

Toric geometry. The assumption in Section 2 that the solution is homogeneous along a fixed axis gives the usual cylindral geometry. In contrast, we can understand the lattice solutions as vortices defined on a torus, as follows. Fix a fundamental cell Ω of the underlying lattice, and identify the opposite sides of this parallelogram. The the gauge-periodic condition in Propositition 2.1 amounts to the appropriate boundary condition on $\partial\Omega$. When the lattice satisfies the *Bradlow condition*

$$v_1 \cdot v_2 \sin\angle(v_1, v_2) > 4\pi N$$

for some (vortex number) $N \in \mathbb{N}$, then one can show that the Bogomolny equations (10) has a large space of solutions whose moduli space is isomorphic to $\mathbb{T}^N/S^N$, the N-fold symmetric product of tori. The coordinates are given by the zeros of the order parameter Ψ as before. See [19, Section 7.14.2].

The cocyle condition. In terms of symmetry action, Proposition 2.1 says that if (Ψ, A) is a λ-lattice, then

$$T_s^{\text{trans}}(\Psi, A) = T_{g_s}^{\text{gauge}}(\Psi, A) \quad (s \in \lambda), \tag{19}$$

where g_s is an appropriate gauge exponent. This equivalence implies an important *cocycle condition* (terminology as in HEP and number theory):

$$g_{s+t}(x) - g_t(x+s) - g_s(x) \in 2\pi\mathbb{Z}. \qquad (s, t \in \lambda) \tag{20}$$

Indeed, on the one hand, by (19) and that translation is commutative, $T_{s+t}^{\text{trans}} = T_{t+s}^{\text{trans}} = T_t^{\text{trans}} T_s^{\text{trans}} = T_{g_t}^{\text{gauge}} T_s^{\text{trans}}$ maps Ψ to $e^{ig_t(x+s)}\Psi(x+s) = e^{ig_t(x+s)}e^{ig_s(x)}\Psi(x)$. On the other hand, $T_{s+t}^{\text{trans}} = T_{g_{s+t}}^{\text{gauge}}$ maps Ψ to $e^{ig_{s+t}(x)}\Psi(x)$. Thus $e^{ig_{s+t}(x)} = e^{ig_t(x+s)}e^{ig_s(x)}$ and the difference in the gauge exponents must be an integer multiple of 2π.

One immediate implication of (20) is that magnetic flux through a fundamental cell Ω of a lattice (Ψ, A) is quantized.

Proposition 2.3 (Quantization of magnetic flux per fundamental cell)**.** *If (Ψ, A) is a λ-lattice solution of* (GL) *with basis v_1, v_2, then*

$$\frac{1}{2\pi}\int_\Omega \operatorname{curl} A =: N \in \mathbb{Z}.$$

Proof. By Stoke theorem we have $\int_\Omega \operatorname{curl} A = \int_{\partial\Omega} A$. Parametrize along the lattice edges, this integral is

$$\int_0^1 v_1 \cdot (A(sv_1 + v_2) - A(sv_1)) - v_2 \cdot (A(v_1 + sv_2) - A(sv_2))\, ds.$$

Since A is λ-equivariant, by (17) this becomes

$$\int_0^1 v_1 \cdot \nabla g_{v_2}(sv_1) - v_2 \cdot \nabla g_{v_1}(sv_2)\, ds.$$

By fundamental theorem of calculus, we have shown

$$\int_\Omega \operatorname{curl} A = g_{v_2}(v_1) - g_{v_2}(0) - g_{v_1}(v_2) + g_{v_1}(0).$$

Next, we show that for every $x \in \mathbb{R}^2$,

$$c(g_s(x)) := \frac{1}{2\pi}(g_{v_2}(x + v_1) - g_{v_2}(x) - g_{v_1}(x + v_2) + g_{v_1}(x) \in \mathbb{Z}.$$

By (20) and that *translational symmetry is commutative*, we have

$$g_{v_2}(x + v_1) + g_{v_1}(x) - g_{v_1+v_2}(x),\ g_{v_1}(x + v_2) + g_{v_2}(x) - g_{v_1+v_2}(x) \in 2\pi\mathbb{Z}.$$

Taking the difference of these two numbers we have the desired result. In particular, at $x = 0$,

$$\frac{1}{2\pi}c(g_s(0)) = \frac{1}{2\pi}\int_\Omega \operatorname{curl} A \in \mathbb{Z}.$$

□

As the above computations have showed, the integer quantity $c(g)$ depends on the function g only. $c(g)$ is the *Chern number* of g and is discussed, for instance, in Manton and Sutcliffe [19, Chapters 3] in a more general setting. Here the number N is the vortex number in Section 2, interpreted as the number of quanta of flux per fundamental cell of a lattice.

Automorphy factor. The cocycle condition (20) is isolated by Tzaneteas and Sigal in [22] to study the stability of lattice solutions under a simple class of perturbation. It is well studied in the context of algebraic geomtry and number theory. There, the function $e^{ig_s(x)}$ is called *automorphy factor*. Two automorphy factors $e^{ig_s(x)}$ and $e^{ig'_s(x)}$ are said to be equivalent if they satisfy $g'_s(x) = g_s(x) + \chi(x+s) - \chi(x)$ for some function χ, and a function Ψ that satisfies $T_s^{\text{trans}}\Psi = e^{g_s(x)}\Psi$ is said to be a $e^{ig_s(x)}$*-theta function*. Gunning in his classification of automorphy factors [11, Theorem 2] shows that every gauge-exponent g_s satisfying (17), (20) is equivalent to

$$\frac{b}{2} s \cdot Jx + c_s, \tag{21}$$

where

$$c_{s+t} - c_s - c_t - \frac{b}{2} s \cdot Jx \in 2\pi\mathbb{Z}. \tag{22}$$

Here $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a symplectic matrix and $b = \frac{1}{|\Omega|}\int_\Omega \operatorname{curl} A = \frac{1}{|\Omega|}c(g_s)$ is the average magnetic flux per fundamental cell. (Direct computation shows the gauge-exponents (21) satisfy the cocycle condition (20).) In Proposition 2.3 below we show how b is related to (the Chern number of) the gauge-exponents. The classification of automorphy factors is through the irreducible representation of the group of lattice translation. In connection to Abrikosov lattice, in [4] Chenn, Smyrnelis, and Sigal use this fact together with bifurcation theory to give a rigorous existential result for lattices with N quanta of lattice per unit cell.

Complexify the plane $\mathbb{R}^2$ by $(x, y) \mapsto x + iy$. Recall that (GL) has Galilean symmetry. So after rotation and translation if necessary, we can assume w.l.o.g. that the underlying lattice is of the form

$$\lambda = r(\mathbb{Z} + \tau\mathbb{Z}) \tag{23}$$

where

$$r := \sqrt{\frac{2\pi N}{b\Im\tau}}, \quad \tau := v_1/v_2 \text{ with } \Im\tau > 0. \tag{24}$$

This gives the shape parameter $\tau \in \mathbb{H}$.

So far the lattices have three parameters: the shape, or the basis v_1, v_2 of λ; the number of quanta of flux per fundamental cell, which equals to the Chern number $N = c(g_s)$ of the gauge exponent; the size of the fundamental cell, $|\Omega|$. one can also choose a gauge so that the gauge-exponents g_s are of the form (21). Using a rescaling, we can also normalize the fundamental cell to have size 2π. We summarize these trimming in the following proposition.

Proposition 2.4. *Let (Ψ, A) be a pair of functions $(\Psi, A) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2$. Define the rescaled fields (ψ, a) by*

$$(\psi, a) := (r\Psi(rx), rA(rx)), \quad r := \sqrt{\frac{|\Omega|}{2\pi}} \overset{(22)}{=} \sqrt{\frac{N}{b}}.$$

Then (Ψ, A) solves (GL) iff (ψ, a) solves

$$\begin{cases} (-\Delta_a - \omega)\psi = -\kappa^2|\psi|^2\psi \quad (\omega := \kappa^2 N/b), \\ \operatorname{curl}^* \operatorname{curl} a = \Im(\bar{\psi}\nabla_a\psi). \end{cases} \tag{25}$$

Moreover, define the rescaled Ginzburg-Landau free energy by

$$\mathcal{E}_\omega(\psi, a) := \frac{1}{2\pi}\int_\omega (|\nabla_a\psi|^2 + |\operatorname{curl} a|^2 + \frac{\kappa^2}{2}(|\psi|^2 - \frac{\omega}{\kappa^2})), \tag{26}$$

where ω is the normalized fundamental cell of

$$\lambda^{normal} := \sqrt{\frac{2\pi}{|\Omega|}}\lambda.$$

Then $|\omega| = 2\pi$, and with the Ginzburg-Landau free energy E_Ω in (1),

$$\frac{1}{|\Omega|}E_\Omega(\Psi, A) = \mathcal{E}_\omega.$$

Moreover, if (Ψ, A) *is a* λ*-lattice satisfies appropriate gauge conditions , then* (ψ, a) *is a* λ^{normal}*-lattice satisfying the analogous equations*

$$\psi(x+s) = e^{i(\frac{N}{2}x\cdot Js + c_s)}\psi(x), \quad a(x+s) = a(x) + \frac{N}{2}Js \quad (s \in \lambda^{normal}), \tag{27}$$

$$\int_\omega (a(x) - \frac{N}{2}Jx) = 0, , \tag{28}$$

$$\operatorname{div} a = 0, . \tag{29}$$

The constants c_s *in* (27) *satisfy the cocycle condition* (22)*.*

Remark. The proof of the rescaling properties of the equation and the energy functional is done by direct computation. Under the complexification (23)-(24), the rescaled lattice depends on the shape parameter τ alone:

$$\lambda^{\text{normal}} = \sqrt{\frac{2\pi}{\Im\tau}}(\mathbb{Z} + \tau\mathbb{Z}).$$

Therefore the space of all normalized lattices is parametrized by the upper half-plane $\mathbb{H}$. This way fields (ψ, a) in (27) depend on N alone. In effect, we have set the average magnetic flux per cell to be $b = N$.

Proposition 2.4 is our frame work for the existential problem of lattices. This is treated using bifurcation theory in Section 5. The result implies Theorem 2.2.

Dynamics of lattices. The dynamical properties of lattices are very similar to those of vortices. As the lattice model pertains mostly to condensed matter physics, here we only consider the first-order dynamics. We summarize some major results below:

(1) in [7], Demoulini and Stuart study the effective dynamics (on the moduli space) for vortices on a generic Riemannian manifold under the Chern-Simons-Schrodinger dynamic. As remarked earlier, the lattice dynaimcs correspond to the special case when the manifold is a flat, two-dimensional torus;
(2) in [22], Sigal and Tzenenas study the stability problems of lattices under the (GE) dynamics. The authors show that for any given underlying lattice λ s.th. $\kappa_c(\lambda) < \kappa$, where $\kappa_c(\lambda) < 1/\sqrt{2}$ is the critical threshold in (18), the corresponding λ-lattice solution is asymptotically stable under gauge-periodic perturbations;
(3) since gauge-periodic perturbations are not common in the applications to superconductivity, in [23] the same authors consider a larger class of finite-enery (H^1) perturbations. They show that a λ-lattice is asymptotically stable if $\kappa > 1/\sqrt{2}$ and certain auxiliary functions are positive.

It is worth noting that the stability results contradict the common belief that only triangular lattices (which minimizes the free energy) with $\kappa > 1/\sqrt{2}$ are stable. The proof of the improved stability result is rather involved. To give an idea of the dynamical stability problem without too much technicality, in Section 6 we prove the second result above.

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