THE SCALING PROCEDURE AND TYPE I SINGULARITY OF THE MEAN CURVATURE FLOW

JINGXUAN ZHANG

The blow-up criterion. In what follows we refer to [Man11, Sig13]. Let $\phi_t : \Sigma^d \to \mathbb{R}^{d+1}$, $t \in [0, T)$ be a family of immersions evolving by the mean curvature flow,

$$\partial_t \phi = H(\phi)\nu(\phi). \tag{1}$$

Write $S_t = \phi_t(\Sigma)$ and suppose S_0 is a closed and orientable. We also write $\phi(\sigma, t) = \phi_t(\sigma)$.

Previously we have shown the *blow-up criterion* for MCF with compact initial condition. We give a quick review. The point is that if we put $f(\sigma, t) := |A(\sigma, t)|^2$ were $A(\sigma, t)$ is the second fundamental form at $\phi_t(\sigma)$, then

$$\partial_t f \leq \Delta f + 2f^2.$$

Put $F(t) = \sup_{\sigma \in \Sigma} f(\sigma, t)$. By the compactness of S_0, Hamilton's Lemma, and the maximum principle, we have

(1) $F(t) > 0$ for all t.
(2) $F(t)$ is locally Lipshitz in time and therefore differentiable a.e..
(3) $\dot{F} \leq 2F^2$ a.e..

By the last differential inequality, we get

$$\frac{1}{F^2}\dot{F} = -\frac{d}{dt}\frac{1}{F} \leq 2.$$

Integrating this over $0 \leq t \leq s < T$, we have

$$\frac{1}{F(t)} - \frac{1}{F(s)} \leq 2(s - t).$$

Now, if $F(s) \to \infty$ as $s \to T$, then the last equation becomes

$$\frac{1}{F(t)} \leq 2(T - t) \quad (t < T).$$

Transposing this, we get the blow-up rate estimate

$$\sup_{\sigma \in \Sigma} |A(\sigma, t)| \geq \frac{1}{\sqrt{2(T - t)}} \quad (t < T). \tag{2}$$

Definition 0.1 (Type I singularity)**.** *Say ϕ_t has a Type I singularity at T if there is $C_0 > 1$ s.th.*

$$\sup_{\sigma \in \Sigma} |A(\sigma, t)| < \frac{C_0}{\sqrt{2(T - t)}}. \tag{3}$$

By (2), Type I singularity means the quantity $\sup_{\sigma \in \Sigma} |A(\sigma, t)| \sim (T - t)^{-1/2}$ as $t \to T^-$.

The most important consequence of Type I singularity is that the any sequence of functions $\phi_n := \phi_{t_n}$ with $t_n \to t$ converges to some limit function ϕ_T, where a priori this limit depends on the chosen sequence. (It is a difficult question to determine wheter this limit ϕ_T is unique.) Indeed, for $0 \leq t \leq s \leq T$ and $\sigma \in \Sigma$, we have

$$\begin{aligned} |\phi_s(\sigma) - \phi_t(\sigma)| &\leq \int_t^s |\partial_t \phi(\sigma, t')| \, dt' \\ &= \int_t^s |H(\sigma, t')| \, dt' \\ &\leq \int_t^s \sqrt{n} |A(\sigma, t')| \, dt' \\ &\leq C_0 \sqrt{2d(T - t)}. \end{aligned} \tag{4}$$

Here since $|A| = \sqrt{\kappa_1^2 + \ldots + \kappa_d^2}$ and $H = \kappa_1 + \ldots + \kappa_d$, we have $|H| \leq \sqrt{n}|A|$ by Cauchy-Schwartz. Uniform estimate (4) shows ϕ_n is Cauchy in $C(\Sigma, \mathbb{R}^{d+1})$ and therefore converges.

Definition 0.2. *Call all points in $\phi_T(\Sigma)$ for any limit ϕ_T the* reachable points.

Huisken's rescaling procedure. The motivation is that if one looks for standing wave (or self-similar) solutions of the form $\phi_t(\sigma) = \lambda(t)\psi(\sigma)$, then using the relation $H(\phi) = H(\lambda\psi) = \lambda^{-1}H(\psi)$ and $\nu(\psi) = \nu(\phi)$, we have

$$\dot{\lambda}\psi = \lambda^{-1}H(\psi)\nu(\psi) \implies H(\psi) = \lambda\dot{\lambda}\psi \cdot \nu(\psi). \tag{5}$$

But ψ is independent of t, so it follows $\lambda\dot{\lambda} = \text{const.}$. Solving this ODE we see that λ must be of the form

$$\lambda = \sqrt{\lambda_0^2 - 2at}, \tag{6}$$

for some $\lambda_0 > 0$ and $a \in R$. Since for standing waves $S_t = \lambda(t)S_0$, we see that

$$\begin{cases} a > 0 \implies \lambda \to 0 \text{ as } t \to T \implies S_0 \text{ is a } \textit{shrinker}. \\ a = 0 \implies S_0 \text{ is an equilibrium of (1)} \implies S_0 \text{ is } \textit{minimal}. \\ a < 0 \implies \lambda \to \infty \text{ as } t \to \infty \implies S_0 \text{ is an } \textit{expander}. \end{cases}$$

If S_0 is compact then by the blow-up criterion (2), we see that T must be finite and only the first case above is possible.

Consider now the time dependent rescaling for a solution $\phi(x,t)$ to (1):

$$\psi(\sigma,\tau) = \lambda^{-1}(t)\phi(\sigma,t) \quad \tau := \int_0^t \lambda^{-2}(\lambda')\, dt'. \tag{7}$$

Differentiating this, we get

$$\partial_t\psi = \partial_\tau\psi\dot{\tau} = -\lambda^{-2}\dot{\lambda}\phi + \lambda^{-1}\partial_t\phi \implies \partial_\tau\psi = H(\psi)\nu(\psi) - \lambda\dot{\lambda}\psi.$$

Here again we use the relation $H(\psi) = \lambda H(\phi)$ and (1). It follows 1. If in (7) the function $\lambda(t) \equiv \lambda > 0$, then we recover the scaling symmetry $\phi(\sigma,t) \to \lambda^{-1}\phi(\sigma, \lambda^{-2}t)$. 2. If in (7) the function $\lambda(t)$ is given by (6), and S_0 is compact, then as we discussed above, $a > 0$ and we can choose $\lambda_0^2 = 2aT$ in (6) where T is the first singular time. Then (7) becomes

$$\psi(\sigma,\tau) = \lambda^{-1}(t)\phi(\sigma,t) \quad \lambda = \sqrt{2a(T-t)}, \quad \tau = -\frac{1}{2a}\ln(T-t). \tag{8}$$

This gives the *rescaled MCF*

$$\partial_\tau\psi = H(\psi)\nu(\psi) + a\psi \quad (a \in \mathbb{R}). \tag{9}$$

By (5), the standing waves are equilibria of (9).

The gradient structure of the rescaled MCF. Previously we have seen MCF (1) as the gradient flow of the area functional

$$\partial_t\phi = -V'(\phi), \quad V(\phi) = \int_\Sigma d\mu. \tag{10}$$

Here μ is the canonical measure induced by the immersion $\phi : \Sigma^d \to \mathbb{R}^{d+1}$. We omit this when there is no ambiguity. We now show that the rescaled flow (9) also has a gradient structure.

Fix $a > 0$ in (8). Put $\rho_a(\sigma) = e^{-a|\phi(\sigma)|^2/2}$ (so that $\rho_a : \Sigma \to \mathbb{R}$) and $V_a := \int_\Sigma \rho\, d\mu$ (the *(Gaussian) weighted measure*).

Proposition 0.3. *For a normal variation $\eta = f\nu$ we have*

$$\partial_\tau\psi = -V_a'(\psi), \quad dV_a(\psi)\eta = -\int_\Sigma (H + a\psi\cdot\nu)\nu\cdot\eta\rho. \tag{11}$$

Proof. Recall the definition of Gâteaux derivative: For a functional $E : M \to \mathbb{R}$ over a (possibly infinite dimensional) Riemannian manifold M, $dE(u)\xi := \partial_s|_{s=0}u_s$ (whenever the latter exists) for a path $u_s \in M$ s.th. $u_0 = u$ and $\partial_s|_{s=0}u_s = \xi$, and $E'(u) \in T_uM$ is defined by the relation $\langle E'(u), \xi\rangle = dE(u)\xi$ for every ξ as above.

Now, consider a family of normal variations ψ_s s.th. $\psi_0 = \psi$ and $\partial_s|_{s=0}\psi_s = \eta$. Since $\eta = f\nu$, we have

$$(\psi\cdot\nu)(\nu\cdot\eta) = (\psi\cdot\nu)f = \psi\cdot f\nu = \psi\cdot\eta.$$

Plugging this into the formula

$$\partial_s|_{s=0}V_a(\psi_s) = \int_\Sigma -H\nu\cdot\eta\rho + \partial_s|_{s=0}\rho(\psi_s) = -\int_\Sigma (H\nu\cdot\eta + a\psi\cdot\eta)\rho$$

gives (11). Here the first term follows from the first variational formula of the area functional. □

The rescaled limit. From the gradient structure (11) we can see that the rescaled flow (9) has translation symmetry. Choose now $a = 1$ and fix a reachable point $p := \phi_T(\sigma_0)$. Put the translated and rescaled flow as

$$\tilde{\psi_\tau} = \frac{\phi - p}{\sqrt{2(T-t)}}.$$

Then the rescaled second fundamental form satisfies

$$|A(\tilde{\psi}(\sigma,\tau))| = \sqrt{2(T-t)}|A(\phi(\sigma,t))| \le C_0 \quad (\sigma \in \Sigma, t < T, \tau < \infty), \tag{12}$$

by the Type I condition (3). This gives an uniform bound on the second fundamental form $|H(\tilde{\psi}(\sigma,\tau)| \le \sqrt{d}C_0$. Lastly, one can compute for

$$|\tilde{\psi}(\sigma_0,\tau)| = |\frac{\phi(\sigma_0,t)-p}{\sqrt{2(T-t}}| \overset{(4)}{\le} C_0\sqrt{d}.$$

These are all immediate consequences of Type I singularity.

In what follows we write $\tilde{S}_\tau$ for the rescaled surface $\tilde{\psi}_\tau(\Sigma)$, and $\rho = \rho_1$. We omit time dependence when there is no ambiguity. We also write $\int_S (\cdot) = \int_\Sigma \psi_*(\cdot)\, d\mu$ for an immersed surface $S = \psi(\Sigma)$.

Proposition 0.4 (Rescaled monotonicity formula)**.**

$$\frac{d}{d\tau}\int_{\tilde{S}_\tau} \rho = -\int_{\tilde{S}_\tau} \rho|H - y\cdot\nu|^2. \tag{13}$$

Proof.

$$\begin{aligned} L.h.s. &= (\frac{d\tau}{dt})^{-1}\frac{d}{dt}\int_S \frac{e^{-\frac{|x-p|^2}{4(T-T)}}}{2(T-t)^{d/2}} \\ &= -2(T-t)\int_S \frac{e^{-\frac{|x-p|^2}{4(T-T)}}}{2(T-t)^{d/2}}|H(x) + \frac{(x-p)\cdot\nu}{2(T-t)}|^2 \quad \text{by Huisken's monotonicity formula} \\ &= -\tau^2\int_{\tilde{S}}\rho|\frac{H(y)}{\lambda} + \frac{y\cdot\nu}{\lambda}|^2 = r.h.s. \end{aligned}$$

Here in the last integral, $y = \frac{x-p}{\tau}$ as in the rescaling $\phi \to \tilde{\psi}$. □

Lemma 0.5 (Stone's lemma)**.** *(1) There is $C = C(d,R,T,V(S_0))$ s.th. $V(\tilde{S}\cap B_R(0)) \le C$.*
(2) There is $C = C(d,T,V(S_0))$ s.th. $\int_{\tilde{S}} e^{-|y|} \le C$.
(3) For every $\epsilon > 0$ there is $R = R(\epsilon,d,T,V(S_0))$ s.th. $\int_{\tilde{S}\setminus B_R(0)} \rho \le \epsilon$.

Proof. 1. The monotonicity formula (13) shows the weighted measure $\int_{\tilde{S}_\tau}\rho \le \int_{\tilde{S}_{-\frac{1}{2}\log T}}\rho$ for all $\tau \ge -\frac{1}{2}\log T$. Write χ_R for the indicator function for $B_R(0)$. Then

$$\begin{aligned} V(\tilde{S}\cap B_R(0)) &= \int_{\tilde{S}_\tau}\chi_R \\ &\le \int_{\tilde{S}_\tau}\chi_R e^{(R^2-|y|^2)/2} \quad \text{since } R^2 - |y|^2 \ge 0 \text{ on the ball } B_R(0) \\ &\le e^{R^2/2}\int_{\tilde{S}_{-\frac{1}{2}\log T}}\rho = e^{R^2/2}\int_{S_0}\frac{e^{-\frac{|x-p|^2}{4T}}}{(2T)^{d/2}}. \end{aligned}$$

2. The point is that $\tilde{\psi}$ solves (9), and therefore

$$\begin{aligned} \frac{d}{d\tau}\int_{\tilde{S}} e^{-|y|} &= \int_{\tilde{S}}(d - H^2 - \frac{1}{|y|}(H\nu + y)\cdot y)e^{-|y|} \\ &\le \int_{\tilde{S}}(C_{d,C_0} - |y|)e^{-|y|} \quad \text{by the uniform estimate on } H(\tilde{\psi}) \\ &= C_{d,C_0}\int_{\tilde{S}}(1 - \frac{|y|}{C})e^{-|y|} \\ &\le C'_{d,C_0}(V(B_C(0)\cap\tilde{S}) - V((\mathbb{R}^{d+1}\setminus B_{2C}(0))\cap\tilde{S})). \end{aligned}$$

The last inequality follows from the decomposition of $\tilde{S}$ into its intersectoin with $B_C(0)$, $B_{2C}(0)\backslash B_C(0)$ and $\mathbb{R}^{d+1}\backslash B_{2C}(0)$, and then throwing away the negative middle term. This estimates shows that either $\frac{d}{d\tau}\int_{\tilde{S}} e^{-|y|} \le 0$, or $V(B_C(0)\cap\tilde{S}) \ge V((\mathbb{R}^{d+1}\setminus B_{2C}(0))\cap\tilde{S})$. In the first case $\int_{\tilde{S}} e^{-|y|} \le \int_{\tilde{S}_{-\frac{1}{2}\log T}} e^{-|y|}$. In the second case the result follows from part (1).

3.Write $S^0 = \tilde{S}\cap B_1(0)$ and $S^k = \tilde{S}\cap\{|y| : 2^{k-1} \le |y| \le 2^k\}$ for $k = 1,2,\ldots$. Then by the part (2), $V(S^k) \le Ce^{2^k}$ and therefore

$$\int_{S^k} e^{-|y|^2/2} \le Ce^{-\frac{1}{2}(2^{k-1})^2}e^{2^k} = Ce^{2^k - 2^{2k-3}}.$$

Now for small ϵ choose large K s.th.

$$\sum_{k\geq K} Ce^{2^k - 2^{2k-3}} < \epsilon.$$

Then the result follows by taking $R = 2^{K-1}$, whence $\int_{\tilde{S}\setminus B_R(0)} \rho = \sum_{k\geq K} \int_{\tilde{S}} e^{-|y|^2/2} < \epsilon$ by construction. □

We can now the consequence of Stone's lemma about the limit surface $\tilde{S}_\infty$, if it exists.

Corollary 0.6. *Suppose there is a sequence $\tau_n \to \infty$ s.th. the sequence of rescaled surfaces $\tilde{S}_{\tau_n}$ converges to some limit surface $\tilde{S}_\infty$ (i.e. the immersions converge locally uniformly). Then $\int_{\tilde{S}_\infty} e^{-|y|} < \infty$.*

Moreover, the limit surface $\tilde{S}_\infty$ is a equilibrium of the rescaled equation (9).

Proof. For every $R > 0$,

$$\int_{\tilde{S}_\infty \cap B_R(0)} e^{-|y|} \leq \liminf_{n\to\infty} \int_{\tilde{S}_{\tau_n} \cap B_R(0)} e^{-|y|} \leq \liminf_{n\to\infty} \int_{\tilde{S}_{\tau_n}} e^{-|y|} \leq C,$$

where C is as in Stone's lemma, part (2).

The limit surface is stationary by the gradient flow structure of (9). Alternatively, integrating the rescaled monotonicity formula (13) we have

$$\int_{-\frac{1}{2}\log T}^{\infty} (\int_{\tilde{S}_\tau} (\rho |H - y\cdot\nu|^2))\, d\tau \leq \int_{\tilde{S}_{-\frac{1}{2}\log T}} \rho < \infty.$$

The finiteness of the l.h.s. integral forces the integrant to vanish as $\tau \to \infty$. □

When does the sequence of rescaled surfaces converge? With some arguments, the uniform bound on the rescaled second fundamental form suffices to show the convergence. One would need to bound the desired norm of the immersion $\tilde{\psi}$ by that of its second fundamental form. Then one uses Arzela-Ascoli to conclude subsequential local convergence.

References

[Man11] Carlo Mantegazza, *Lecture notes on mean curvature flow*, Progress in Mathematics, vol. 290, Birkhäuser/Springer Basel AG, Basel, 2011. MR2815949

[Sig13] Israel Michael Sigal, *Lectures on mean curvature flow, https://www.math.toronto.edu/sigal/courses.html*, 2013 (accessed November, 2020).