The conception of Riemannian geometry

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Euclid's five Postulates

Euclid's *Elements* (c. 300 BC) is based on his five *postulates*:

P1. To draw a straight line from any point to any point.

P2. To produce a finite straight line continuously in a straight line.

P3. To describe a circle with any center and radius.

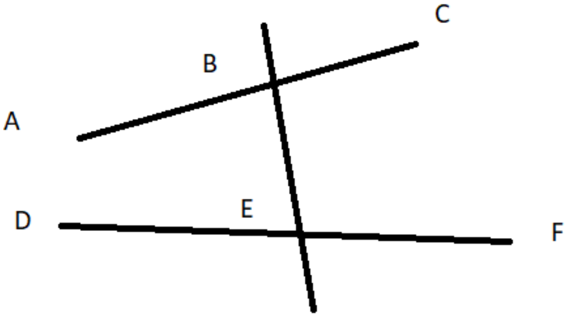
P4. That all right angles equal one another.

P5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

One can sort these postulates using contemporary theory:

1. As topological space: P1 (linear space), P2 (continuity).
2. As metric space: P3 ($\exists$ a binary function associate pair of points to a definite distance).
3. As Riemannian manifold: P4 (homogeneity), P5 ($\iff$ zero curvature).

But of course this classification is not available to early geometers.



If $\vec{BA}$ and $\vec{ED}$ are extended, then they intersect.

Before Riemann's solution

Recorded efforts to cut P5 off can be found even among the early scholors

> *This [PP] ought to be struck from the postulates altogether. For it is a theorem... and requires for its demonstration a number of definitions as well as theorems. And the converse of it is proved by Euclid himbself as a theorem. (Proclus, 412-485)*

The mentioned theorem, Prop. I.17 in the *Elements* (sum of any two angles in a triangle $< \pi$) is actually independent of PP ($\iff$ angle sum of a triangle $= \pi$).

Attempts of proof remain unfruitful: they are either incorrect or based on additional hypotheses. Major breakthroughs are made by Sacherri, Lambert and Legendre (c. 1730), which leads to the works of Gauss, Bolyai, Lobachevsky (c. 1830).

An important development in this century is: the former group of mathematicans all started with the conviction that **geometry without PP is absurd**. E.g. The title of Sacherri's *Euclid Freed of Every Flaw)*. The latter generation however found a meaningful theory of **geometry without PP plausible**.

EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipſa univerſæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinenſi Univerſitate Matheſeos Profeſſore.
OPUSCULUM
EX.MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.

Ex Typographia Pauli Antonii Montani. *Superiorum permiſſu.*

EUCLID
FREED OF EVERY FLECK
OR
A GEOMETRIC ENDEAVOR IN WHICH ARE
ESTABLISHED THE FOUNDATION
PRINCIPLES OF UNIVERSAL
GEOMETRY

BY
GIROLAMO SACCHERI
OF THE SOCIETY OF JESUS
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF PAVIA.

A WORK DEDICATED TO
THE NOBLE SENATE OF
MILAN BY THE AUTHOR

MILAN, 1733
PAOLO ANTONIO MONTANO SUPERIORUM PERMISSU

Title page of Fr. Saccheri's treatise

Gauss

Entering 19th century, mathematicians become less inclined to committ to prove PP. Gauss, in particular, eventually believed this would be impossible:

> *I come more and more to the conviction that the necessity of our geometry cannot be proved at least not by the human intellect nor for the human intellect. (Gauss to Olbers, April 28th, 1817)*

From his private letters to his colleagues, Gauss seems to have already conceived of *a seperate notion* from a given abstract space, which determines certain geometric properties that are hitherto attributed to the space itself, or *completely a priori*. The property of parallel, as stated in PP, is among such properties indeterminate basing on the space alone.

I have consolidated many things further and my conviction that we cannot justify geometry completely a priori has if possible grown even stronger. (Gauss to Bessel, Jan 27th, 1829)

Auch über ein anderes Thema, das bei mir schon fast 40 Jahr alt ist, habe ich zuweilen in einzelnen freien Stunden wieder nachgedacht; ich meine die ersten Gründe der Geometrie; ich weiss nicht, ob ich Ihnen je über meine Ansichten darüber gesprochen habe. Auch hier habe ich manches noch weiter consolidirt, und meine Ueberzeugung, dass wir die Geometrie nicht vollständig a priori begründen können, ist wo möglich noch fester geworden. Inzwischen werde ich wohl noch lange nicht dazu kommen, meine sehr ausgedehnten Untersuchungen darüber zur öffentlichen Bekanntmachung auszuarbeiten, und vielleicht wird diess auch bei meinen Lebzeiten nie geschehen, da ich das Geschrei der Boeoter scheue, wenn ich meine Ansicht ganz aussprechen wollte. — Seltsam ist es aber, dass ausser der bekannten Lücke in Euklid's Geometrie, die man bisher umsonst auszufüllen gesucht hat, und nie ausfüllen wird, es noch einen andern Mangel in derselben gibt, den meines Wissens niemand bisher gerügt hat, und dem abzuhelfen keineswegs leicht (obwohl möglich) ist. Diess ist die Definition des Planum als einer Fläche, in der die irgend zwei Puncte verbindende gerade Linie ganz liegt. Diese Definition enthält mehr, als zur Bestimmung der Fläche nöthig ist, und involvirt tacite ein Theorem, welches erst bewiesen werden muss.

Mit grossem Vergnügen habe ich in Berlin gehört, dass schon damals an Ihrer Abhandlung über Ihre Pendelversuche gedruckt wurde. Ich erwarte die Erscheinung dieser Arbeit, die alles frühere so weit hinter sich lässt, mit wahrer Sehnsucht.

Die ganze Zeit meines Aufenthalts in Berlin ist für mich äusserst genussreich gewesen und hat auch manche meiner Vorstellungen von dieser Stadt und Ihrem Staate überhaupt berichtigt.

Leben Sie wohl, lieber Bessel, und erfreuen Sie bald einmal mit Nachrichten von Sich

Ihren

C. F. Gauss.

The cited letter, from the Archive of Academy of Sciences and Humanities, Göttingen

A contemporary of Gauss, Russian mathematician Lobachevsky came to the conclusion that PP itself cannot be derived as a theorem, but rather an empirical property that one chooes to attach to the description of space.

N.I. Lobachevsky (1792-1856)

It is well known that in geometry the theory of parallels has so far remained incomplete. The futile efforts from Euclid's time on throughout two thousand years have compelled me to suspect that the concepts themselves do no contain the truth which we have wished to prove, but that it can only be verified like all other physical laws by experiment, such as astronomical observation. (Lobachevsky, 1829)

On the Hypotheses

Riemann, a doctoral student of Gauss, contemplated the possibility and consequences of there being no proof of PP. His idea along this line is best summarized in his habilitation lecture in front of the Faculty at Göttingen in 1854, *On the Hypotheses Which Lie at the Bases of Geometry*.

G.F.B. Riemann (1826-66)

In this posthumously published essay, which later become widely influential, Riemann outlined his vision of a separated notion of metric from that of the underlying space, the foundation of what is now known as *his* Geometry.

The central idea: the pair (M, g) where $M =$ the "multiply extended magnitude" (i.e. underlying n-dimensional manifold), and $g =$ the "measure relation" (i.e. a metric tensor on M) together, instead of M itself, determine the geometric properties of a space.

. . .A multiply extended magnitude is capable of different measure relations. . . The propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience. (Plan of the Investigation)

Preliminaries

For Riemann, the notion of *manifold* includes both the one we are familiar today, as well as that of discrete set.

> *According as there exists among these specialisations [elements in a set] a continuous path from one to another or not, they form a continuous or discrete manifoldness: the individual specialisations are called in the first case points, in the second case elements, of the manifoldness. (Sec.I.1)*

Riemann regards the “discrete manifold” as the prevelant case in real life, and a theory of such can be build upon “the postulate that certain given things are to be regarded as equivalent”. (Of course, modern set theorists may not agree on this point.) The only real life example of continuous manifold that Riemann can think of are “the positions of perceived objects and colours”. Other than that, “More frequent occasions for the creation and development of these notions occur first in the higher mathematic.” (Sec I.1)

Construction of manifolds

After setting up these notions, Riemann goes on to give a constrution of "multiply extended manifoldness", viz. an n-dimensional manifold, my means of iterated parametrization.

If in the case of a notion whose specialisations [viz. points, elements] form a continuous manifoldness, one passes from a certain specialisation in a definite way to another, the specialisations passed over form a simply extended manifoldness, ... If one now supposes that this manifoldness in its turn passes over into another entirely different, and again in a definite way, namely so that each point passes over into a definite point of the other, then all the specialisations so obtained form a doubly extended manifoldness . . . and it is easy to see how this construction may be continued (Sec.I.2).

Reduction of manifolds

Next, Riemann descibed how to obtain an n-dimensional manifold from an $(n+1)$-dimensional one via level set (assuming $\operatorname{Sing} M = \emptyset$, so that all level sets are exactly one dimension less.), "the reduction of determinations of place in a given manifoldness to determinations of quantity". From this procedure Riemann would "make clear the true character of an n-fold extent"

. . . [L]et us take a continuous function of position within the given manifoldness, which, moreover, is not constant throughout any part of that manifoldness. Every system of points where the function has a constant value, forms then a continuous manifoldness of fewer dimensions than the given one. These manifoldnesses pass over continuously into one another as the function changes. . . the cases of exception (the study of which is important) may here be left unconsidered. Hereby the determination of position in the given manifoldness is reduced to a determination of quantity and to a determination of position in a manifoldness of less dimensions. (Sec. I.3)

Determination of metric tensor

Next, to determine the desired properties of the "measure relation", viz. the metric tensor g, Riemann points out the necessity of introducing purely analytic means in addition to mere geometric considerations.

> *...[T]he measure relations of which such a manifoldness is capable, and of the conditions which suffice to determine them ... can only be studied in abstract notions of quantity, and their dependence on one another can only be represented by formulæ. (Sec. II)*

Consider two points point $p, q \in \mathbb{R}^d$ and a C^1 parametrization $x : [0, 1] \to M$ s.th. $x(0) = p, x(1) = q$. Riemann's postulates for admissible "measure relations", viz. a (infinitesimal) metrics, denoted ds, are as follow (Sec. II.1):

1. "[T]he length of lines is independent of their position, and consequently every line is measurable by means of every other.' ' $\iff$ the choice of ds itself does not depend on p, q.
2. "[A]n expression which will thus contain the quantities x and the quantities dx. " $\iff$ ds is a function of x, $\dot{x}$.
3. "[I]f all the quantities dx are increased in the same ratio, the linear element will vary also in the same ratio." $\iff$ ds is homogeneous in $\dot{x}$

Let $M = \mathbb{R}^d$. Fix $p \in M$. Denote $ds = ds|_p$ the metric element at p to be determined. Riemann considers a C^2 function $f : M \to \mathbb{R}$ s.th. $f(p) = 0$, and $f(q)$ increases as $d(p, q)$ increases. Since p is a minimum, Riemann further assumes the Hessian of f at p is strictly positive. He then reasons:

This differential expression, then, of the second order [i.e. the 2-form $\partial_{ij} f dx^i dx^j$] remains constant when ds remains constant, and increases in the duplicate ratio when the dx, and therefore also ds, increase in the same ratio [as postulated above]; it must therefore be ds^2 multiplied by a constant, and consequently ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx, in which the coefficients are continuous functions of the quantities x. For Space, when the position of points is expressed by rectilinear coordinates, $ds = \sqrt{\sum (dx)^2}$. (Sec.II.1)

Riemann concludes that in general,

$$ds^2|_x = \partial_i \partial_j f(x, dx) dx^i dx^j$$

for some $f \in C^2(\mathbb{R}^d)$. In modern terms, the metric tensor is given by $ds^2 = g_{ij} dx^i dx^j$, and the familiar postulates on g follow from the properties of f above:

1. $g_{ij} = g_{ji}$: since $f \in C^2$, this follows from Clairaut's rule
2. $g_{ij} v^i v^j > 0$ for non-zero v: since the Hessian of f is positive definite.

Other possibilities are acknowledged, but later dismissed on the ground of less geometric interpretation.

> *I restrict myself, therefore, to those manifoldnesses in which the line-element is expressed as the square root of a quadric differential expression. (Sec. II.1)*

Implicitly, Riemann favours quadratic expression because Pythagorean identity is satisfied at the infinitesimal scale (i.e. ds^2 is quadratic in dx^i) .

The necessity of g

The necessity to specify a metric tensor is then given by a consideration on the degree of freedom.

$\frac{1}{2}n(n+1) =$ the number of free coefficients in a symmetric 2-tensor g_{ij}, as in a quadratic expression for line segment $ds^2 = g_{ij}dx^i dx^j$.

$n =$ the number of coordinates, considered as substitutions $y_i = y_i(x_1, \ldots, x_n), i = 1, \ldots, n$ of a given coordinate x.

Clearly $\frac{1}{2}n(n+1) > n$, so there are free coefficients independent of coordinization. These must therefore be intrinsic in geometric description.

Based on the number of free coefficients in the metric, Riemann distinguish flat and curved manifolds:

> *Manifoldnesses in which, as in the Plane and in Space, the line-element may be reduced to the form $\sqrt{\sum(dx)^2}$, are therefore only a particular case of the manifoldnesses to be here investigated... [T]hese manifoldnesses in which the square of the line-element may be expressed as the sum of the squares of complete differentials I will call* **flat**. *(Sec. II.1)*

For a general curved manifold, the remaining $\frac{1}{2}n(n-1)$ degree of freedom are found by specifying the sectional curvatures at each point, the number of which equals to that of the linearly independent two-dimensional subspaces of the n-dimensional tangent space at that point: this number $= C(n, 2) = \frac{1}{2}n(n-1)$. E.g. For surface in 3-space, $\frac{1}{2}n(n-1) = 1$, and sectional curvature $\equiv$ Gauss curvature.

Modern interpretation: the symmetries of Riemann curvature $R_{ijkl}(p)$ make it completely determined once the sectional curvatures $K(\Pi)$ are known for all linearly independent $\Pi \subset T_pM$. Then one can recover g_{ij} from the definition of sectional curvature

$$K(\Pi) = K(v, w) = \frac{R_{ijkl} v^i v^j w^k w^l}{(g_{ik} g_{jl} - g_{ij} g_{kl}) v^i v^j w^k w^l}$$

$$(\Pi = \mathrm{span}(v, w) \subset T_pM).$$

Analytic v. geometric quantities

For Riemann, the sectional curvatures K are purely **geometric** quantities, in contrast to the purely **analytic** quantities g_{ij} appeared in the "measure relations". Once the (sectional) curvatures are brought to the front, Riemann start a discussion of constant *(sectional)* curvature surfaces (not CMC!).

Surfaces of constant (sectional) curvatures

On a constant curvature surface, since the metric tensor is determined by the curvatures, the former also remains constant. The metric element is then given by

$$ds = \sqrt{g_{ij} dx^i dx^j} = \frac{1}{1 + \frac{\alpha}{4}\sqrt{(x^i)^2}} \sqrt{\sum (dx^i)^2} \quad (\alpha = \text{curvature}).$$

This is actually the only real formula in the whole lecture.

Riemann listed some special properties on surface of constant curvature (Sec. II.5):

1. "If we regard these surfaces as *locus in quo* [v. "site in which", or ambience] for surface regions moving in them, as Space is *locus in quo* for bodies, the surface regions can be moved in all these surfaces without stretching": $\forall p, q \in M$ with constant curvatures, $\exists$ neighbourhoods $U \ni p, V \ni q$ in M and an isometry $\phi : U \to V$ s.th. $\phi(U) = V$. This ϕ is the "movement"
2. "The surfaces with positive curvature can always be so formed that surface regions may also be moved arbitrarily about upon them without bending": this follows the fact that surfaces of constant positive curvature can be realized as spheres.

3 “[B]ut not those with negative curvature.”

4 “[I]n surfaces of zero curvature [there is] also an independence of direction from position, which in the former surfaces does not exist.”: The tangent spaces T_pM, T_qM are related by parallel transport. The latter in the flat case is independent of the choice of connecting curve joing p, q, unlike for general g.

From empirical observation, free and rigid mobility of bodies appears to be a natural condition on the geometric model of space. Riemann's contemporary Helmholtz postulates this point in his *Über die tatsc̈hlichen Grundlagen der Geometrie ("About the factual basis of Geometry)"* as an axiom. From here Helmholtz derives the constancy of curvature is necessary for the description of space, which in fact is a rather strong restriction.

Application to 3-space

So far, Riemann has developed an abstract theory involving a pair (M, g) of an underlying manifold and an admissible metric. The next problem is to take $M = \mathbb{R}^3$ and consider the various possibilities of g, as well as how to choose which particular g for the description of the physical world.

This theme occurs earlier to Lobachevsky, who wrote:

> *[S]pace, in itself, for itself alone, does not exist for us. Accordingly, there can be nothing contradictory for our understanding if we allow that some forces in nature follow one, others another special geometry. (Lobachevsky, 1829)*

Riemann, as his cited predecessor above, retains that the geometry ($\iff$ the metric tensor g) is determined by empirical observations.

For instance, one can determine the metric g as follows:

1. "[T]he metric properties of space are determined if the sum of the angles of a triangle is always equal to two right angles.": by Gauss-Bonnett, this implies $K \equiv 0$ and this in turn implies g is everywhere flat.
2. "[I]f we assume with Euclid not merely an existence of lines independent of position, but of bodies also, it follows that the curvature is everywhere constant". : Locally isometric $\iff$ constant curvature, in which g is given by the previous formula.

In the study of description of space, Riemann stresses the distinction between what in modern terms amount to topological and metric properties (Sec.III.2)

> *[W]e must distinguish between unboundedness and infinite extent, the former belongs to the extent relations, the latter to the measure relations. . . . The unboundedness of space possesses . . . a greater empirical certainty than any external experience. But its infinite extent by no means follows from this; on the other hand if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value.*

“Unbonudedness” means $\partial M = \emptyset$, a topological property.

“infinite extent” means diam $M = \sup_{p,q \in M} d(p,q) = \infty$, a metric space property.

E.g. in the above paragraph, $\mathbb{R}^3$ is unbounded yet $(\mathbb{R}^3, \delta_{ij})$ has infinite extent. Every surface of constant positive curvature has finite extent (since they can be realized as spheres).

Riemann considers the task of determining the metric as a doomed effort if one were to seek absolute accuracy, since “the possible cases form a continuous manifoldness, every determination from experience remains always inaccurate” (Sec.III.2).

Meanwhile, Riemann restricts his attention at the differential level, while dismissing the “questions about the infinitely great“ as “useless”.

> *But this [the futility of studying at large scale] is not the case with the questions about the infinitely small. It is upon the exactness with which we follow phenomena into the infinitely small that our knowledge of their causal relations essentially depends (Sec.III.3).*

This naturally leads Riemann to use techniques of differential geometry, recently developed by his doctoral advisor Gauss.

Riemann maintains a position similar to Lobachevsky, in that empirical observation determines what amounts to a fair choice of g. Yet Riemann would not to go so far as did Helmholtz in stipulating rigid movement as a part of axiom in his theory. Indeed, Riemann has in mind of a much wider range of possibility for the metric tensor g. The only restrictions come from analytic consideration.

Now it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena (Ibid).

Conclusion

Riemann's greatest contribution to geometry is his clear articulation of a novel structure of space, wherein the "measure relation" is separated from the "n-ply extended manifoldness". For Riemann, the central object of differential geometry is the metric tensor g, instead of the underlying space M. The fomer contains essentially all the information regarding the geometric properties.

To apply his theory of geometry to physics, one must determine g suitably from empirical observation, even if the result thus derived is against intuition. The point of building a mathematical frame work for what we now call Riemannian geometry, is to remove the restriction imposed by convention and intuition that reduces one's view to the Euclidean case. This then will allow physicist to obtain better vision of the understanding of empirical facts.

Epilogue

The answer to these questions [i.e. determining the geometric description of space] can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices. (Sec.III.3)

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