Geometric concepts in the Ginzburg-Landau theory of superconductivity

Jingxuan Zhang

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Ojectives

I would like to introduce the following concepts:

1. The structure of the Ginzburg-Landau free energy.
2. Mixed state in superconductivity.
3. Connection to codimension-2 minimal submanifold

Set up of the problem

Let $\Omega \subset \mathbb{R}^d$ be a domain. The *Ginzbug-Landau free energy* is given by

$$E_\Omega(\Psi, A) := \frac{1}{2} \int_\Omega \left\{ |\nabla_A \Psi|^2 + |\operatorname{curl} A|^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}. \quad (1)$$

Here $\kappa > 0$ is a material constant, $\Psi(x) : \Omega \to \mathbb{C}$ the *order parameter*, $A(x) : \Omega \to \mathbb{R}^d$ the *vector potential*, and $\nabla_A := \nabla - iA$ the covariant derivative (on the line bundle $\Omega \times \mathbb{C}$) that couples A to Ψ. The physically possible states are the minimizers of (1)

The two homogeneous states

The purely superconducting state is given by

$$\Psi_s \equiv 1, \qquad A_s \equiv 0.$$

This solution reflects the Meissner effect that characterizes superconductivity: the expulsion of the external magnetic field from the bulk of the superconducting material.

The normal, non-superconducting state is given by

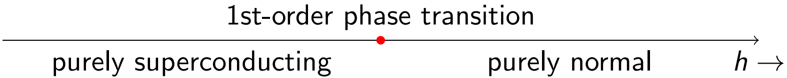
$$\Psi_0 \equiv 0, \qquad \operatorname{curl} A_0 \equiv H,$$

where $H : \Omega \to \mathbb{R}^3$ equals to the applied field (not shown in (1)).

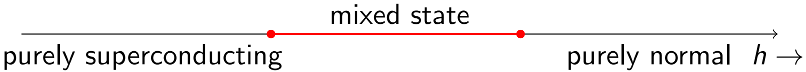
Type I and type II superconductors

Consider constant applied field $H(x) \equiv h \geq 0$. The material parameter κ determines the the behaviour of the superconductor as the applied field strength h varies.

Type I ($\kappa^2 < 1/2$)



Type II ($\kappa^2 > 1/2$)



Mixed state in 2D: magnetic vortices I

Let $\Omega = \mathbb{R}^2$. For Type II material, there exist mixed states of the form

$$\Psi_N = f_N(r) e^{iN\theta}, \quad A_N = a_N(r) \nabla (N\theta), \tag{2}$$

where (r, θ) is the polar coordinate of $x \in \mathbb{R}^2$, $f_N(0) = 0$ and rapidly increseas to 1 away from $r = 0$, and N is an integer. These solutions are called N-*vortices*. The N-vortices describe the mixed states with N quanta of magnetic flux trapped in where the order parameter Ψ_N vanishes.

Mixed state in 2D: magnetic vortices II

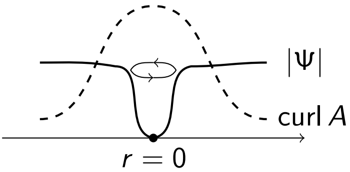


Figure above shows a cross section of a vortex solution near a core at $r = 0$, where the superconducting electron density $|\Psi|$ vanishes and the magnetic field curl A penetrates. For an N vortex, the order parameter Ψ winds around the center N times, and the penetrating field has N quanta of magnetic flux.

Mixed state in 3D: magnetic filament

Let $\Omega \subset \mathbb{R}^3$. Let $S \subset \Omega$ be a codimension-2 submanifold. At the critical coupling $\kappa^2 = 1/2$, there exist mixed state Ψ_1, A_1 s.th. the rescaling $\Psi_\epsilon(x) = \Psi_1(\epsilon x), A_\epsilon(x) = A_1(\epsilon x)$ satisfy

$$|\nabla_{A_\epsilon}\Psi_\epsilon|^2 + |\operatorname{curl} A_\epsilon|^2 + \frac{1}{4}(|\Psi|^2 - 1)^2 dx \to dH^{d-2}|_S \quad (\epsilon \to 0).$$

Here $|\Psi_\epsilon| = 0$ on the submanifold S and rapidly increase to 1 away from S (at the length scale ϵ).

Open question: how about $\kappa^2 > 1/2$?

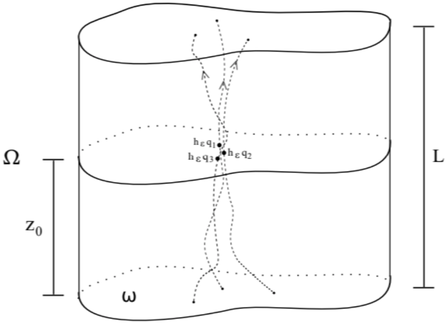


Figure: Several filaments in a cylindrical domain $\Omega = \omega \times L \subset \mathbb{R}^3$.

(Contreras and Jerrard, Geom. Funct. Anal. 27 (2017), no. 5, 1161-1230)

In this picture, as $\epsilon \to 0$, the energy density of $\Psi_\epsilon, A_\epsilon$ tends to the weight measure along the appx. vertical curves in Ω.

Geometric aspect: connection to geometric flow

Consider now the various dynamics for GL, the simplest one being the gradient flow

$$\partial_t u = -E'_\Omega(u), \quad u = (\Psi, A).$$

As one scales down the length unit as in the static filament example, one can show that this equation converges uniformly in time to the gradient flow of the codim-2 area functional.
The latter is the generalized mean curvature flow. This follows from variational argument (Bethuel-Orlandi-Smets '09).

An important open problem is to consider Schrödinger type dynamics for GL,

$$\partial_t u = -\zeta E'_\Omega(u), \tag{3}$$

where $\zeta = \mathrm{diag}(\mu, \sigma)$ for some fixed complex number μ and invertible matrix σ.

Conjecture

Let $\Omega \subset \mathbb{R}^3$. For $\mu = i|\log \kappa|$ and invertible σ, the equation (3) converges (*in a suitable sense*) to the binormal curvature flow $\partial_t \gamma = \partial_s \gamma \times \partial_{ss} \gamma$, where $\gamma(s)$ parametrizes the concentration curve.

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Robert Jerrard from Univesity of Toronto will give an online talk on March 31 on this subject as a part of our group seminar. Details to follow.