Some results on adiabatic approximation for nonlinear evolutions

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Setting

Consider an abstract evolution equation

$$\partial_t u = JE'(u). \tag{1}$$

- $u = u_t \in U$ is a C^1 path in some open set U in a real Hilbert space X.
- $E : U \subset X \to \mathbb{R}$ is a sufficiently regular functional.
- $J = -1$, or $J^* = -J$ (i.e. symplectic)

Depending on J, (1) is either a gradient flow or Hamiltonian equation.

Motivation: the concentration problem

Consider the following problem:

- $X =$ a suitable space of functions from $\Omega \subset \mathbb{R}^{n+k} \to \mathbb{R}^k$,
- $u_0 =$ some *localized* configuration, f.ex. outside a neighbourhood of some n-dimensional concentration set $\sigma_0 \subset \Omega$, all derivatives of u_0 vanish rapidly.
- u_t remains localized near some concentration set σ_t.

Question: What kind of geometric flow governs the motion of σ_t?

Example: multi-vortex dynamics

Consider the time dependent Ginzburg-Landau (GL) equations

$$
\begin{aligned}
&\psi : \mathbb{R}^2 \to \mathbb{C}, \\
&\partial_t \psi = JE'(\psi), \\
&E(\psi) = E^\epsilon(\psi) := \int_{\mathbb{R}^2} \frac{1}{2}|\nabla \psi|^2 + \frac{1}{4\epsilon^2}\left(|\psi|^2 - 1\right)^2, \\
&J = -1 \text{ for gradient flow, or } -i \text{ for dispersive dynamics.}
\end{aligned}
\tag{2}
$$

Originally from condensed matter theory.

- $\psi =$ wave function of electronic or Bose-Einstein condensate.
- $0 < \epsilon \ll 1$ is a small length scale.

For the GL dynamics (2),

- $X = H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$

  (NB. Finite energy configurations are not in $L^2(\mathbb{R}^2)$);

- $\psi^{(1)}(x)$ = radially symmetric steady state (1-vortex):

  $\left|\psi^{(1)}\right| = 0$ at $x = 0$, and $\left|\psi^{(1)}\right| \to 1$ rapidly away $x = 0$;

- $u_0 := \prod_{j=1}^{n} \psi^{(1)}(x - z_j(0))$;

- $Z_0 = (z_j(0))$ widely separated, u_t generated by $u_0 \implies$

  u_t remains localized at $Z_t = (z_j(t))$.

Question: How does Z_t evolve?

For the dispersive case $J = -i$, it was shown in (Ovchinnikov-Sigal 98) that the concentration points $Z = Z_t \in \mathbb{R}^{2n}$ evolves as

$$\partial_t Z = \mathcal{J}\mathcal{E}'(Z), \quad \mathcal{E}(Z) \sim \sum_{i \neq j} \log |z_i - z_j|.$$

Here $\mathcal{J}$ is a symplectic $2n$-matrix.

For the gradient flow $J = -1$, it was shown in (Bethuel-Orlandi-Smets 07) using very different method that

$$\partial_t Z = -\mathcal{E}'(z).$$

Note that the energy property of the effective dynamics agrees with that of the full dynamics.

Adiabatic approximation

In this study we propose an abstract scheme to find effective dynamics for the abstract evolution (1), known in the physics literature as the method of *adiabatic approximation*.

Main assumption: there exists $f : \Sigma \subset Y \to M \subset U$ where

- Σ is a C^1 manifold in a Hilbert space Y;
- f is a smooth immersion;
- M consists of some low energy configurations (precisely conditions given below), not necessarily steady states.

We call Σ the *moduli space*, and M the space of *approximate solitons*.

The adiabatic approximation scheme goes roughly as follows:

1. Find a nonlinear projection from a tubular neighbourhood M_δ of M to Σ;
2. Find an evolution $\partial_t \sigma = F(\sigma)$ for a path $\sigma_t \in \Sigma$, s.th. if u_t solves (1) and remains in M_δ, and σ_t is associated to u_t as above, then σ_t evolves according to this equation;
3. $u_0 \in M_\delta \implies u_t \in M_{\sqrt{\delta}}$ for a long (but possibly finite) time.

In short:

$$\text{adiabatic decomposition} \to \text{effective equation} \to \text{validity.}$$

Approximate solitons

Precise conditions for elements in M are:

$$\left\| E'(v) \right\|_X \leq \epsilon, \tag{C1}$$

$$L_v := E''(v) \text{ is self adjoint and } L_v \Big|_{(JT_vM)^\perp} \geq \alpha > 0, \tag{C2}$$

$$L_v \Big|_{JT_vM} \leq \epsilon. \tag{C3}$$

- (C1) says every $v \in M$ is approximate critical point.
- (C2)-(C3) are stability conditions.
- In general, $\alpha = \alpha(\epsilon) \to 0$ as $\epsilon \to 0$.

To explain the terminology for M and Σ, note in the limiting case,

1. M consists of exact critical points of $E \iff \left\|E'(v)\right\|_X = 0$.
2. M consists of solitons (steady states generated by symmetry breaking, or Goldstone bosons), L_v has a spectral gap at 0, $\Sigma =$ broken symmetries of E, and $f =$ the action of Σ on M $\implies$ (C2) holds and

$$L_v\Big|_{JT_vM} = 0.$$

Step 1: adiabatic decomposition

Lemma

There is $\delta > 0$ *s.th. for evert* $u \in X$ *and* $\text{dist}(u, M) < \delta$*, there exists* $\sigma \in \Sigma$ *s.th.*

$$\left\| u - f(\sigma) \right\|_X = \inf \left\{ \|u - v\|_X : v \in M \right\}. \tag{3}$$

Proof.

1. For each fixed σ, use Implicit Function Theorem to find a projection from $B_\delta(f(\sigma)) \subset X \to \Sigma$.
2. Show δ can be made independent of σ (essential for later use)

□

Step 2: effective dynamics-preliminaries

Fix coordinate spaces Y, X for $T_\sigma\Sigma$, $T_v M$ respectively.

Write $g_\sigma : Y \to X$ for the action of $df(\sigma) : T_\sigma\Sigma \to T_{f(\sigma)}M$.

Let g_σ^* be the adjoint to g_σ.

Define

$$\begin{array}{rcccl} \mathcal{J}_\sigma & : & Y & \longrightarrow & Y \\ & & \xi & \longmapsto & g_\sigma^* J^{-1} g_\sigma \xi \end{array},$$

$\mathcal{E} : \Sigma \to \mathbb{R}$ is the pull-back of E by f.

Properties of the operator $\mathcal{J}_\sigma$

1. J is a symplectic operator $\Longrightarrow$ so is $\mathcal{J}_\sigma$ on the tangent bundle $T\Sigma$, since

$$\langle \mathcal{J}_\sigma \xi, \xi \rangle = \langle g_\sigma^* J^{-1} g_\sigma \xi, \xi \rangle = \langle J^{-1} g_\sigma \xi, g_\sigma \xi \rangle = 0 \quad (\xi \in T_\sigma \Sigma).$$

2. g_σ is injective $\Longrightarrow$ $\mathcal{J}_\sigma$ is invertible.

Thus if J is symplectic, then $\mathcal{J}_\sigma$ induces a nondegenerate symplectic form on $T\Sigma$.

Various uniform estimates

We assume the map f satisfies for some $0 < \alpha < 1$ that

$$\|g_\sigma\|_{Y \to X} \sim \epsilon^{-\alpha}. \tag{U1}$$

- Natural assumption when $f(\sigma)$ describes a perturbation of some stable interface. (shown later for the application).
- (U1) implies

$$\|\mathcal{J}_\sigma\|_{Y \to Y} \leq C\epsilon^{-2\alpha}, \tag{U2}$$

$$\left\|\mathcal{J}_\sigma^{-1}\right\|_{Y \to Y} \leq C\epsilon^{2\alpha}, \tag{U3}$$

$$\left\|\mathcal{E}'(\sigma)\right\|_{Y} \leq C\epsilon^{1-\alpha}. \tag{U4}$$

Step 2: effective dynamics-statement

Lemma

Let u_t be a solution to (1). Suppose $\mathrm{dist}(u_t, M) < \delta$ *for some* $0 \leq \delta \ll 1$. *Write $u_t = v_t + w_t$ with $v_t = f(\sigma_t)$, where σ_t is given by the adiabatic decomposition. Then there is $C > 0$ in dependent of time s.th.*

$$\left\| \partial_t \sigma - \mathcal{J}_\sigma^{-1} \mathcal{E}'(\sigma) \right\|_Y \leq C \epsilon^{1+\alpha} \|w\|_X \tag{4}$$

Remark

1. This lemma suggests that the effective dynamics governing the motion ofσ_t of a flow near M is given by

$$\partial_t \sigma = \mathcal{J}_\sigma^{-1}\mathcal{E}'(\sigma) + \text{lower order terms.} \tag{Eff}$$

2. The definite volume of the projection neighbourhood in Step 1 is used here: this allows the flow to fluctuate in $\mathrm{dist}(u_t, M) < \delta$, for fixed δ independent of t.

Step 2: effective dynamics-proof

1. Expand (1) as

$$\partial_t v + \partial_t w = J(E'(v) + L_\sigma w + N_\sigma(w)), \tag{5}$$

where L_σ is the linearized operator at $v_t = f(\sigma_t)$, and $N_\sigma(w)$ defined by this equation.

Let $Q_\sigma : X \to X$ be the linear projection onto the tangent space $T_{f(\sigma)}M$. Applying Q_σ to both sides of (5), we have

$$\partial_t v - Q_\sigma J E'(v) = Q_\sigma(J L_\sigma w - \partial_t w + J N_\sigma(w)). \tag{6}$$

2. The following identities follow readily from the chain rule:

$$\partial_t v = g_\sigma \, \partial_t \sigma, \quad g_\sigma^* E'(f(\sigma)) = \mathcal{E}'(\sigma).$$

Using these, we get the identity

$$\mathcal{J}_\sigma^{-1} g_\sigma^* J^{-1} (\partial_t v - Q_\sigma J E'(v)) = \partial_t \sigma - \mathcal{J}_\sigma^{-1} \mathcal{E}'(\sigma).$$

Thus by the uniform estimates for g_σ^* and $\mathcal{J}_\sigma^{-1}$, we have

$$\left\| \partial_t \sigma - \mathcal{J}_\sigma^{-1} \mathcal{E}'(\sigma) \right\|_Y \leq C \epsilon^\alpha \left\| \partial_t v - Q_\sigma J E'(v) \right\|_X . \qquad (7)$$

Conclusion: the error in (Eff) is controlled by the l.h.s. of (6).

3. Consider now the r.h.s. of (6)

$$Q_\sigma(JL_\sigma w - \partial_t w + JN_\sigma(w)).$$

These three terms can be bounded respectively as follows:

$$\|Q_\sigma JL_\sigma w\|_X \leq C\epsilon\|w\|_X \,, \tag{8}$$

$$\|Q_\sigma \partial_t w\|_X \leq C\|\partial_t \sigma\|_Y\|w\|_X \,, \tag{9}$$

$$\left\|Q_\sigma JN_\sigma(w)\right\|_X \leq C\|w\|_X^2 \,. \tag{10}$$

To derive these, we use the uniform bounds $\|Q_\sigma\|_{X\to X} \leq C$, (U1)-(U4), and the fact that $w_t \in (JT_v Q)^\perp = \ker Q$.

4. Combining (6)-(10), we have

$$\left\| \partial_t \sigma - \mathcal{J}_\sigma^{-1} \mathcal{E}'(\sigma) \right\|_Y \le C \left(\epsilon^\alpha \|\partial_t \sigma\|_Y \|w\|_X + \epsilon^{1+\alpha} \|w\|_X \right) \quad (11)$$

It remains to control $\|\partial_t \sigma\|_Y$. Two cases:

1. $\|\partial_t \sigma\|_Y < \left\| \mathcal{J}_\sigma^{-1} \mathcal{E}'(\sigma) \right\|_Y \implies \|\partial_t \sigma\|_Y \le C\epsilon^{1+\alpha}$ by (U3)-(U4).
2. Otherwise, if $\|\partial_t \sigma\|_Y \ge \left\| \mathcal{J}_\sigma^{-1} \mathcal{E}'(\sigma) \right\|_Y$, then for $\|w\|_X \le \delta = \delta(\epsilon) \ll 1$, (11) implies

$$\|\partial_t \sigma\|_Y \le C\epsilon^{1+\alpha} \left(1 + \|w\|_X\right). \quad (12)$$

In either case, r.h.s. of (11) is bounded by $C\epsilon^{1+\alpha}\|w\|_X$. □

Step 3: validity-statement

Theorem

There is $0 \leq \delta \ll 1$ *with the following property: Let* $u_0 \in X$ *be an initial configuration s.th.* $\mathrm{dist}(u_0, M) < \delta$. *Let* u_t *be the flow generated by* u_0 *under (1).*

Then for all $t \leq T$ *where*

$$\begin{cases} T = \infty & \text{if } J = -1 \text{ in } (1), \\ T = O(\epsilon^{-\alpha}) & \text{if } J \text{ is symplectic in } (1), \end{cases}$$

the flow u_t *remains* ϵ*-close to* M.

Remark

By Step 2, if u_t is uniformly ϵ-close to M with sufficiently small ϵ, then

$$\partial_t \sigma = \mathcal{J}_\sigma^{-1} \mathcal{E}'(\sigma) + O(\epsilon^{2+\alpha}). \tag{Eff}$$

Since $\left\| \mathcal{J}_\sigma^{-1} \mathcal{E}'(\sigma) \right\|_Y = O(\epsilon^{1+\alpha})$ by (U3)-(U4), this shows that the remainder in (Eff) is of lower order.

Proof for gradient flow

1. In Step 2, we have shown

$$\left\| \partial_t \sigma - \mathcal{J}_\sigma^{-1} \mathcal{E}'(\sigma) \right\|_Y \leq C \epsilon^{1+\alpha} \|w\|_X \tag{13}$$

Claim: For all $t > 0$,

$$\mathrm{dist}(u_t, M) < \epsilon. \tag{14}$$

By the continuity of the flow, for $\delta \ll \epsilon$, claim (14) holds at least up to some small $T_1 > 0$. For $t \leq T_1$, we have (**??**) by (13). Thus it remains to show this T_1 is large.

2. Strategy: Find a differential inequality for the quadratic form $\frac{1}{2}\langle L_\sigma w, w\rangle$, which accounts for most of the energy dissipation. Then (**??**) follows from the coercivity condition (C2).

Remark

1. This quadratic form is a Lyapunov-type functional.
2. To simplify writing, we assume $\|w\|_X \leq 1$, an ansatz that we justify at the end of the proof.

3. Compute

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}\langle L_\sigma w, w\rangle}{\mathrm{d}t} &= \langle \partial_t w, L_\sigma w\rangle + \frac{1}{2}\langle(\partial_t L)w, w\rangle \\
&= \langle -\partial_t v - (E'(v) + L_\sigma w + N_\sigma(w)), L_\sigma w\rangle + \frac{1}{2}\langle(\partial_t L)w, w\rangle
\end{aligned} \tag{15}
$$

Using the stability properties (C2)-(C3), we find some $\beta > 0$ independent of t s.th.

$$
\frac{1}{2}\left|\frac{\mathrm{d}\langle L_\sigma w, w\rangle}{\mathrm{d}t}\right| \le C\left(\|\partial_t \sigma\|_Y + \|w\|_X - \beta\right)\|w\|_X^2 + \epsilon\|w\|_X . \tag{16}
$$

4. Since L_σ is uniformly bounded, by (16), we can find $\gamma > 0$ s.th.

$$(\partial_t + \gamma)\langle L_\sigma w, w\rangle \leq C(\|\partial_t \sigma\|_Y + \|w\|_X - \beta/2)\|w\|_X^2 + \epsilon\|w\|_X \,. \quad (17)$$

Ansatz:

$$\|\partial_t \sigma\|_Y + \|w\|_X \leq \beta/2, \quad (18)$$

This will be justified later

By this ansatz, we can drop the first term in the r.h.s. of (17), and multiply both side by an integration factor $e^{\gamma t}$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\gamma t}\langle L_\sigma w, w\rangle\right) \leq C\epsilon e^{\gamma t}\|w\|_X \,. \quad (19)$$

5. Integrating (19), and then dividing the integration factor, we find

$$\begin{aligned}\langle L_\sigma w, w\rangle &\leq C\left(e^{-\gamma t}\langle L_{\sigma_0} w_0, w_0\rangle + \epsilon M(t)\right)\\ &\leq C\left(e^{-\gamma t}\|w_0\|_X^2 + \epsilon M(t)\right) \quad \left(M(t) := \sup_{t'\leq t}\|w\|_X\right).\end{aligned} \tag{20}$$

Together with the coercivity (C2), we find

$$M(t) \leq C\left(e^{-\gamma t}\|w_0\|_X + \epsilon\right).$$

Choosing $\delta < \epsilon$, we can conclude from here that $M(t) < C\epsilon$ for all time. For ϵ sufficiently small, the ansatze are satisfied. □

Proof for Hamiltonian system

1. Claim: for $t \leq T = O(\epsilon^{-\alpha})$ and $\delta \leq \epsilon$,

$$\text{dist}(u_t, M) < \epsilon. \tag{21}$$

The main ingredient in the following arguments is the conservation of E along (1) with symplectic J.

2. Consider the expansion

$$E(u) = E(v+w) = E(v) + \langle E'(v), w\rangle + \frac{1}{2}\langle L_\sigma w, w\rangle + R_\sigma(w), \quad (22)$$

where $R_\sigma(w)$ is the super-quadratic remainder.

By condition (C2), (22) implies

$$\|w\|_X^2 \leq C(E(v+w) - E(v) - \langle E'(v), w\rangle - R_\sigma(w)). \qquad (23)$$

3. Since $E(u)$ is conserved along (1), we have

$$E(v+w) = E(v_0+w_0) = E(v_0)+\langle E'(v_0), w_0\rangle+\frac{1}{2}\langle L_\sigma w_0, w_0\rangle+R_\sigma(w_0).$$

Plugging this into (23), and using the approximate critical point property of the approximate solitons, we find

$$\|w\|_X^2 \leq C\alpha^{-1}\left(E(v_0) - E(v) + \epsilon\|w\|_X + \|w\|_X^3 + \epsilon\|w_0\|_X + \|w_0\|_X^2\right). \tag{24}$$

4. It remains to control the energy fluctuation

$$\left| E(v_0) - E(v) \right| .$$

Differentiate the energy $E(t) = E(f(\sigma_t))$, we have

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \langle E'(v), Q_\sigma JE'(v)\rangle + \langle E'(v), Q_\sigma(JL_\sigma w - \partial_t\, w)\rangle \\ &\quad + \langle E'(v), Q_\sigma JN_\sigma(w))\rangle. \end{aligned} \tag{25}$$

We find the three inner products satisfy the estimates

$$\langle E'(v), Q_\sigma JE'(v)\rangle = 0, \tag{26}$$

$$\left|\langle E'(v), Q_\sigma(JL_\sigma w - \partial_t\, w)\rangle\right| \le C\left(\epsilon^2 \|w\|_X + \epsilon^\alpha \|w\|_X^2\right), \tag{27}$$

$$\left|\langle E'(v), Q_\sigma JN_\sigma(w))\rangle\right| \le C\epsilon \|w\|_X^2 . \tag{28}$$

5. Combining (25)-(28) and integrating from 0 to t, we have

$$\left|E(v(t)) - E(v(0))\right| \le Ct\left(\epsilon^2 M(t) + \epsilon^\alpha M(t)^2\right), \qquad (29)$$

where $M(t) := \sup_{t' \le t} \left\|w(t')\right\|_X$.

Plugging(29) into (24), and then dividing both side by $M(t)$, we have

$$M(t) \le \gamma\left(t\left(\epsilon M(t) + \epsilon^2 M(t) + \epsilon^\alpha M(t)^2\right) + \left(\epsilon + \delta + \delta^2\right)\right), \qquad (30)$$

for some $\gamma > 0$ independent of ϵ and δ.

6. Let $T_1 > 0$ be the first time s.th. $\|w\|_X \geq \epsilon$, and recall we have chosen $\delta \leq \epsilon$.

If

$$T_1 < \frac{1}{2}\gamma^{-1}\epsilon^{-\alpha},$$

then we have by (30) that

$$\epsilon < \epsilon + O(\epsilon^{2-\alpha}).$$

For ϵ sufficiently small, this is impossible. Thus claim (21) hold up to a large time $T = O(\epsilon^{-\alpha})$ as desired. □

Remark

Compare the statement of the validity theorem (A) to dynamical stability (B).

(A) A flow started near an approximate soliton stays close to some approximate solitons, not necessarily related by symmetries.

(B) A flow started near a fixed exact soliton stays close to the manifold generated by the symmetries acting on this given soliton.

Application: vortex filament dynamics

To illustrate the adiabatic method, consider the dispersive Ginzburg-Landau equation on bounded $\Omega \subset \mathbb{R}^3$,

$$\frac{\partial\psi}{\partial t} = JE'(\psi),$$

$$E(\psi) = E_\Omega^\epsilon(\psi) := \int_\Omega \frac{1}{2}|\nabla\,\psi|^2 + \frac{1}{4\epsilon^2}\left(|\psi|^2 - 1\right)^2,$$

$$J : \psi \mapsto -i\psi.$$

Equivalently, the full dynamics is

$$i\frac{\partial\psi}{\partial t} = -\Delta\,\psi + \frac{1}{\epsilon^2}\left(|\psi|^2 - 1\right)\psi. \qquad \text{(GL)}$$

Concentration

Similar to planar point vortex, for $\epsilon \ll 1$, any low energy configuration must concentrate near 1D zero set.

Goal: Show the effective dynamics for (GL) is

$$\frac{\partial \gamma}{\partial t} = \mathcal{J} L'(\gamma), \tag{31}$$

known as the binormal curvature flow.

- $\gamma = \gamma_t(s)$ is a C^1 path of curves parametrized by arclength;
- $L(\gamma) = \int |\partial_s \gamma|^2$, $L'(\gamma) = -\partial_{ss} \gamma$;
- $\mathcal{J} := -\partial_s \gamma \times$.

Preliminaries

Take $\Omega := \left\{(x, z) : x \in \omega \subset \mathbb{R}^2, z \in I := [0, 1]\right\}$.

Put

$$X^s := H^s(\Omega, \mathbb{C}), \quad Y^k := \mathbb{R} \times C^k(I, \mathbb{R}^2) \quad (s \in \mathbb{R}, k \in \mathbb{N}).$$

Write elements in Y^k as $\sigma = (\lambda, \gamma)$. X^s, Y^k are real Hilbert spaces with the inner products

$$\langle \psi, \psi' \rangle_X = \int_\Omega \Re(\overline{\psi}\psi'), \quad \langle \sigma, \sigma' \rangle_Y = \int_I \gamma \cdot \gamma' + \mu\mu'. \tag{32}$$

X^1 is the ambient space for the flow (GL)

Approximate filaments

Write

$$C^k_{\text{per}} := \left\{\gamma \in C^k(I,\omega) : \gamma(0) = \gamma(1)\right\}$$

for $k \in \mathbb{N}$. Define

$$\Sigma := \mathbb{R} \times \left(\text{Emb}(I,\omega) \cap C^2_{\text{per}}\right), \quad \Sigma_\delta := \left\{(\lambda,\gamma) \in \Sigma : \|\gamma\|_{Y^2} < \delta\right\}.$$

We view Σ as a submanifold in Y.

Consider the map

$$\begin{array}{rccc} f & : & \Sigma_\delta & \longrightarrow & X \\ & & \sigma = (\lambda,\gamma) & \longmapsto & e^{i\lambda}\psi_\gamma(x,z) \end{array}, \qquad (33)$$

$$\psi_\gamma(x,z) := \psi^{(1)}(x - \gamma(z)).$$

Key properties:

- f is a C^1 immmersion;
- f parametrizes a submanifold $M \subset X^0$ by Σ_δ, with tangent space $T_{f(\sigma)}M = df(\sigma)(T_\sigma \Sigma_\delta)$;
- Can trivialize $T_{f(\gamma)}M \to X^0$;
- Σ, M are Riemannian manifolds with the inner products given in (32).

Call elements in M the *approximate filaments*: since each configuration concentrates near some curve around $\{0\} \times I$.

Asymptotics for the planar vortex

Write $\psi^{(1)} = f(r)e^{i\theta}$ in polar coordinate. Then

$$f \in C^{\infty},\ f' > 0 \text{ for } r > 0, \text{ and} \tag{34}$$

$$f \sim 1 - \frac{\epsilon^2}{2r^2} \quad (r \to \infty), \quad f \sim \frac{r}{\epsilon} - \frac{r^3}{8\epsilon^3} \quad (r \to 0), \tag{35}$$

$$E_\omega(\psi^{(1)}) \leq \pi|\log\epsilon| + C(\omega), \tag{36}$$

$$\left\|\psi^{(1)}\right\|_{L^\infty(\omega)} \leq 1, \tag{37}$$

$$\left\|\nabla\psi^{(1)}\right\|_{L^\infty(\omega)} \leq \frac{C(\omega)}{\epsilon}, \tag{38}$$

$$\left\|\nabla\psi^{(1)}\right\|_{L^2(\omega)} \leq C(\omega)|\log\epsilon|^{1/2}, \tag{39}$$

$$\frac{1}{\epsilon^2}\int_\omega \left(\left|\psi^{(1)}\right|^2 - 1\right)^2 \leq C(\omega). \tag{40}$$

Properties of the approximate filaments

Assume the material parameter $\epsilon \ll 1$ in (GL).

Since $f := |\psi^{(1)}|$ (resp. $g := \left|\nabla \psi^{(1)}\right|$) is strictly increasing (resp. decreasing) sufficiently away from $r = 0$, using the classical asymptotics

$$f \sim 1 - \frac{\epsilon^2}{2r^2} \quad (r \to \infty)$$

we have control over the oscillation of f and g as

$$\left|f(r) - f(s)\right| \leq C\frac{\epsilon^2}{R^2}, \quad \left|g(r) - g(s)\right| \leq C\frac{\epsilon}{R} \quad (r > s \geq R \gg 0), \tag{41}$$

Picture

Let $0 < \alpha < 1$ be given s.th. the planar domain ω contains the ball of radius $1 + \epsilon^{\alpha}$. Then for $\gamma \in \Sigma_{\epsilon^{\alpha}}$,

$$\begin{aligned}
\int_{\Omega} |\psi_\gamma|^2 &= \int_I \int_\omega |\psi_\gamma|^2(x,z) \\
&\leq \int_I \left(\int_\omega |\psi^{(1)}|^2(x) + \int_\omega \left(|\psi_\gamma|^2(x,z) - |\psi^{(1)}|^2(x) \right) \right) \\
&\leq \int_I \left(\int_\omega |\psi^{(1)}|^2(x) + \|\gamma\|_{Y^0} \operatorname{diam}(\omega) \sup_{r > s \geq 1} \left| f(r) - f(s) \right|^2 \right) \\
&\leq \left\| \psi^{(1)} \right\|^2_{L^2(\omega)} + C(\omega) \epsilon^{4+\alpha}
\end{aligned}$$

Picture

Therefore we have

$$\left\|\psi_\gamma\right\|_{X^0} = \left\|\psi^{(1)}\right\|_{L^2(\omega)} + O(\epsilon^{2+\alpha/2}). \tag{42}$$

Similarly, one can show that

$$\left\|\nabla_x \psi_\gamma\right\|_{X^0} = \left\|\nabla_x \psi^{(1)}\right\|_{L^2(\omega)} + O(\epsilon^{1+\alpha/2}). \tag{43}$$

Using these we can show uniform estimate (C1)

$$\left\|E'(f(\sigma))\right\|_{X^0} \leq C\epsilon^{\alpha} |\log \epsilon|^{1/2}$$

for $\|\sigma\|_Y < \epsilon^\alpha$.

Proof.

Compute

$$E'(f(\sigma)) = e^{i\lambda}\left(\nabla_x \psi_\gamma \cdot \gamma_{zz} - \nabla_x^2 \psi_\gamma \gamma_z \cdot \gamma_z\right).$$

For $\|\gamma\|_{C^2} \ll 1$, the leading term is $\nabla_x \psi_\gamma \cdot \gamma_{zz}$. Estimate this:

$$\begin{aligned}\left\|\nabla_x \psi_\gamma \cdot \gamma_{zz}\right\|_{X^0} &\leq 2\left\|\nabla_x \psi_\gamma\right\|_{X^0}\|\gamma_{zz}\|_{C^0} \\ &\leq 2\left(\left\|\nabla_x \psi^{(1)}\right\|_{L^2(\omega)} + C(\omega)\epsilon^{1+\alpha/2}\right)\|\gamma\|_{C^2} \\ &\leq C(\omega)\epsilon^{\alpha}|\log\epsilon|^{1/2}.\end{aligned}$$

Here we use the $\gamma \in \Sigma_{\epsilon^\alpha}$, and the classical estimate

$$\left\|\nabla \psi^{(1)}\right\|_{L^2(\omega)} \leq C(\omega)|\log\epsilon|^{1/2}. \quad \square$$

Stability properties

We have shown that approximate filaments are approximate critical points for the GL energy functional. This is the condition (C1) in the generic scheme.

The stability properties (C2)-(C3) are given as follows.

In the statements,

- $L_\sigma = E''(f(\sigma))$;
- $\|\sigma\|_{Y^2} < \delta \ll 1$;
- Q_σ is the projection onto $T_{f(\sigma)}M$.

Lemma (uniform bound of L_σ)

The operator L_σ is bounded on X^1, and there is $0 < C < \infty$ independent of σ and ϵ s.th.

$$\|L_\sigma\|_{X^1 \to X^0} < C\epsilon^{-1}. \tag{44}$$

$$\|L_\sigma Q_\sigma\|_{X^1 \to X^0} \leq C\delta^{1/2}. \tag{45}$$

Lemma (coercivity)

There is $\alpha = O(|\log \epsilon|^{-1}) > 0$ s.th.

$$\langle L_\sigma \phi, \phi \rangle \geq \alpha \|\phi\|_{X^0} \quad (\phi \in \ker Q_\sigma)\,. \tag{46}$$

Properties of the Fréchet derivative of f

For $\sigma = (\lambda, \gamma) \in \Sigma_{\epsilon^\alpha}$ and $(\mu, \xi) \in Y^k$,

$$df(\sigma)(\mu, \xi) = e^{i\lambda}(i\mu\psi_\gamma - \nabla_x \psi_\sigma \cdot \xi). \tag{47}$$

Note this involves gradient term of the interface.

The map $df(\sigma)$ is uniformly bounded in σ as an operator from $Y^0 \to X^0$, satisfying

$$\left\| df(\sigma)(\mu, \xi) \right\|_{X^0} \leq C(\Omega) |\log \epsilon|^{1/2} \|\sigma\|_{Y^0} . \tag{48}$$

Using (48) we can get uniform estimates

$$\left\| df(\sigma)^* \right\|_{X^0 \to Y^0} \leq C |\log \epsilon|^{1/2}, \tag{49}$$

$$\| \mathcal{J}_\sigma \|_{Y^k \to Y^k} \leq C(\omega) |\log \epsilon|^2, \tag{50}$$

$$\left\| \mathcal{J}_\sigma^{-1} \right\|_{Y^k \to Y^k} \leq C(\omega) |\log \epsilon|^{-2}. \tag{51}$$

for every $k \in \mathbb{N}$.

(48)-(51) correspond to (U1)-(U4). The key property (48) is generic for interface in various phase transition models.

The effective dynamics

So far, all the assumptions of the general theory are verified.

The key components to verify the assumptions are the uniform estimates for ψ_γ,

$$\left\|\psi_\gamma\right\|_{X^0} = \left\|\psi^{(1)}\right\|_{L^2(\omega)} + O(\epsilon^{2+\alpha/2}),$$
$$\left\|\nabla_x \psi_\gamma\right\|_{X^0} = \left\|\nabla_x \psi^{(1)}\right\|_{L^2(\omega)} + O(\epsilon^{1+\alpha/2}).$$

These allows us to translate classical estimates for $\psi^{(1)}$ for the approximate filaments.

Effective dynamics for (GL)

Theorem (effective dynamics)

For any $\epsilon > 0$*, there are* $\delta_1, \delta_2 \ll \epsilon$ *s.th. the following hold:*

Let $u_0 \in X^1$ *be an initial configuration s.th.* $\mathrm{dist}_{X^1}(u_0, M) < \delta_2$*.*

Then there exists $T = O(1/\epsilon)$ *s.th. for* $t \leq T$*,*

$$\mathrm{dist}_{X^1}(u_t, M) = o(\sqrt{\delta_1}). \tag{52}$$

Moreover, for $t \leq T$*, the evolution of* $v_t = f(\sigma_t)$ *evolves as*

$$\partial_t \vec{\gamma}_t(s) = \vec{\gamma}_s \times \vec{\gamma}_{ss} + o_{\|\cdot\|_{C^0}}(\delta_1) \quad \left(|\partial_s \vec{\gamma}| = 1\right), \tag{53}$$

$$\partial_t \lambda = o(\delta_1). \tag{54}$$

Remark

1. The initial data is assumed to be a filament with uniformly small curvature.
2. In (53), the spatial curve $\vec{\gamma}$ parametrizes the set $\{(\gamma(z), z)\} \subset \Omega$ by arclength. The correspondence $\gamma \leftrightarrow \vec{\gamma}$ is one-to-one.
3. The evolution (54) governs the modulation parameter λ, which is not physically relevant.