Geotop presentation on the adiabatic theory for the area-constrained Willmore flow

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1 Introduction

Let (M, g) be a 3-dimensional, complete, oriented Riemannian manifold with non-negative curvature. Consider the area-constrained Willmore (ACW) flow,

$$\partial_t x^N = -W(x) - \lambda H(x). \tag{1}$$

Here, for $t \geq 0$, $x = x_t : \mathbb{S} \to M$ is a family of embeddings of spheres (with orientation compatible with that on M). $\partial_t x^N$ denotes the normal velocity at x, given by $\partial_t x^N := g(\partial_t x, \nu)$, where $\nu = \nu(x)$ is the unit normal vector to Σ at x. $H(x)$ denotes the mean curvature scalar at x. $W(x) := \Delta H(x) + H(x)(\mathrm{Ric}_M(\nu, \nu) + |\mathring{A}|^2(x))$ is the Willmore operator, where $\mathring{A}(x)$ denotes the traceless part of the second fundamental form. λ is the Lagrange multiplier, arising due to the area constraint.

1.1 Configuration spaces and the geometric structure of ACW

In this subsection, we layout the geometric structure of ACW flow (1).

Let $c \gg 1, k \geq 4$ be given. Define the configuration space

$$X^k := H^k(\mathbb{S}, M), \quad X^k_c := \left\{ x \in X^k : |x(\mathbb{S})| = c \right\}. \tag{2}$$

Here, for a surface $\Sigma := x(\mathbb{S}) \subset M$, we denote $|\Sigma| := \int_\Sigma d\mu^g_\Sigma$ the area of Σ, where μ^g_Σ is the area form induced by the embedding x and background metric g. One can check easily that (1) is well-defined in X^k_c. The spaces in (2) are equipped with the L^2-inner product

$$\langle \phi,\, \phi' \rangle := \int_{\mathbb{S}} \langle \phi,\, \phi' \rangle_{\text{Euclidean}} \quad (\phi, \phi' \in X^k). \tag{3}$$

Let $x \in X^k$ and write $\Sigma = x(\mathbb{S})$. The tangent spaces to x at X^k and X^k_c are respectively given by

$$T_x X^k = X^k, \tag{4}$$

$$T_x X^k_c = \left\{ \phi \in T_x X^k : \int_\Sigma H g(\phi, \nu) = 0 \right\}. \tag{5}$$

Here, (5) is due to the well-known first variation formula of the area functional. Notice that, slightly abusing notation, in (5) we view ϕ as a vector field over Σ. With (3), we have a formal Riemannian structure on the configuration spaces X^k and X^k_c.

With this geometric structure of X^k, one can view the equation (1) as the L^2-gradient flow, restricted to X^k_c, of the Willmore energy

$$\mathcal{W}(\Sigma) = \frac{1}{4} \int_\Sigma H^2 \, d\mu^g_\Sigma. \tag{6}$$

We call (the images of) static solutions to (1) *surfaces of Willmore type*, following the nomenclature in [4]. Using Sobolev inequalities, one can show that for $k \geq 4$, the functional $\mathcal{W}$ is well-defined and C^2 (in the sense of Fréchet derivatives) on X^k_c.

Let $d\mathcal{W}(x) : T_x X^k_c \to T_x X^{k-4}_c$ be the Fréchet derivative of $\mathcal{W}$ at an embedding x in the class X^k_c. Define the normal L^2-gradient $\nabla^N \mathcal{W}(x)\phi := d\mathcal{W}(x)\phi$ for every normal, area-preserving variation ϕ on the surface $\Sigma = x(\mathbb{S})$. (This operator ∇^N depends on x.) Then by the first variation formula of the

Willmore energy (see e.g. [1, Sec. 3]), this $\nabla^N \mathcal{W}(x)$ is given by the r.h.s. of (1). This allows us to rewrite (1) as

$$\partial_t x^N = \nabla^N \mathcal{W}(x) \quad (x \in X^k_c).$$

Equivalently, (1) is the (negative) gradient flow of the Hawking mass,

$$m_{\text{Haw}}(\Sigma) := \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \frac{1}{2}\int_\Sigma H^2 \, d\mu^g_\Sigma\right), \tag{7}$$

in the sense that a flow of surfaces evolving according to (1) increases the mass m_{Haw}. For interests from physics related to this problem, especially in general relativity, see [5].

1.2 Main result

Under suitable assumptions on the background manifold, we derive the following results in [6]:

Theorem 1 (Main)**.** *Let $k \geq 4$, $c \gg 1$. Let X^k be the configuration space defined in* (2)*. Fix $R \gg 1$, $\delta \ll 1$ in Definition 1.*

Then there exists a map

$$\tilde{\Phi} : M' := \mathbb{R}_{>R} \times B_1(0) \subset \mathbb{R} \times \mathbb{R}^3 \to X^k$$

with the following property: Let $x_{r,z} := \tilde{\Phi}(r, z)$. There hold:

1. *(Critical point) x_z parametrizes a surface of Willmore type if and only if z is a critical point of the function $\mathcal{W} \circ \tilde{\Phi} : \mathbb{R}^4 \to \mathbb{R}$, restricted to the submanifold $\left\{(r, z) \in M' : \left|\tilde{\Phi}(r, z)(\mathbb{S})\right| = c\right\}$.*
2. *(Stability) Suppose x_z parametrize an admissible surface of Willmore type. Then x_z is uniformly stable with small area-preserving H^k-perturbation* [1] *if z is a strict local minimum of the function $\mathcal{W} \circ \tilde{\Phi}$ restricted to the submanifold $\left\{(r, z) \in M' : \left|\tilde{\Phi}(r, z)(\mathbb{S})\right| = c\right\}$.*
3. *(Effective dynamics) Let $\Sigma_* = x_*(\mathbb{S})$ be an admissible surface. Let $\Sigma_t = x_t(\mathbb{S})$ be the global solution to* (1) *with initial configuration $\Sigma|_{t=0} = \Sigma_*$ as in Theorem 2.*

   Then there exist $\alpha > 0$, $T = O(R^{-\alpha})$, and a path $(r_t, z_t) \in M'$, such that for every $t \geq T$,

   $$\|\tilde{\Phi}(r_t, z_t) - x_t\|_{X^k} = O(R^{-3}). \tag{8}$$

   Moreover, the path (r_t, z_t) evolves according to

   $$\dot{z} = \frac{1}{4\pi}\nabla_z \mathcal{W} \circ \tilde{\Phi}(r, z) + O(R^{-3}), \tag{9}$$

   $$\dot{r} = 4R^{-2} + O(R^{-3}). \tag{10}$$

   In (9) *the leading term is of the order $O(R^{-2})$.*
4. *Conversely, if $(r_t, z_t) \in M'$ is a flow evolving according to* (9)*-*(10)*, then there exists a global solution x_t to* (1) *such that* (8) *holds for this choice of (r_t, z_t) and every $T \leq t \leq T + R$.*

2 Setup of the problem

Below we give the geometric assumptions and preliminary results in Theorem 1.

[1] This means that if y is another admissible surface that is H^k-close to x_z, then for every $\epsilon > 0$ there exists $T > 0$ such that $\|x_t - y_t\|_{X^k} < \epsilon$ for all $t \geq T$, where x_t, y_t are respectively the flows generated by x_z, y under (1).

2.1 Asymptotically Schwarzschild manifold

A 3-dimensional complete Riemannian manifold (M, g) is said to be C^k-close to Schwarzschild if the following holds:

1. $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B_1(0)}$ for some compact subset $K \subset M$.
2. The metric g splits as $g_S + h$, where

$$g_S := \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}$$

is the Schwarzschild metric with ADM mass $m > 0$, and $h \in C^k$ is a small perturbation satisfying

$$h_{ij} = h_{ji}, \quad \partial^\alpha h_{ij} \leq \eta |x|^{-(2+|\alpha|)} \quad (|\alpha| \leq k,\ |x| \gg 1), \tag{11}$$

for some fixed small decay coefficient $\eta \ll 1$. Here $x \in \mathbb{R}^3$ denotes the coordinate in the asymptotic chart on M.

Physically, for applications to general relativity, such manifold M is a perturbation of the static Schwarzschild black hole $(\mathbb{R}^3 \setminus \overline{B_{m/2}(0)}, g_S)$.

To simplify notations, throughout the paper we normalize ADM mass to be $m = 2$. We assume that the ambiance space M is C^k-close to Schwarzschild for sufficiently large k, and that in (11) the decay coefficient $\eta \ll 1$. Thus in what follows we take

$$(M, g) = (\mathbb{R}^3 \setminus \overline{B_1(0)}, g_S + h)$$

where h is as in (11).

To use results in [2–4], we assume the scalar curvature Sc on M satisfies the following decay properties:

$$x^j \partial_{x^j} \left(|x|^2 \operatorname{Sc}\right) = o(|x|^2), \tag{12}$$

$$\operatorname{Sc}(x) - \operatorname{Sc}(-x) = o(|x|^4). \tag{13}$$

The asymptotically-flatness condition (12) is satisfied if g is C^k-close to Schwarzschild with $k \geq 4$, and $\operatorname{Sc} = o(|x|^4)$, in which case $\nabla \operatorname{Sc} = o(|x|^{-5})$. (13) means the scalar curvature on M is asymptotically even. Geometrically, condition (12) provides quantitative control for various estimates involving extrinsic geometric quantities.

2.2 Preliminary results

Let $R \gg 1$ be given. Let $K \subset M$ be a large compact set. As explained in the last subsection, for asymptotically Schwarzschild manifold M, we can identify the $M \setminus K$ with its coordinate space $\mathbb{R}^3 \setminus \overline{B_R(0)}$.

Let $\delta > 0$ be given.

Definition 1 (Admissible surfaces)**.** For a closed surface $\Sigma \subset M$, define the inner and outer radii $\rho(\Sigma)$, $\lambda(\Sigma)$ as

$$\rho(\Sigma) = \min_{x \in \Sigma} |x|, \tag{14}$$

$$\lambda(\Sigma) := \sqrt{|\Sigma| / 4\pi}. \tag{15}$$

We say Σ is admissible if the interior of Σ contains K, and

$$\rho(\Sigma) > R, \quad \left|\frac{\rho(\Sigma)}{\lambda(\Sigma)} - 1\right| + \int_\Sigma |\mathring{A}|^2 < \delta. \tag{16}$$

Here $\mathring{A}$ denotes the traceless part of the second fundamental form on Σ.

Remark 1. Geometrically, a surface Σ is admissible if the origin lies sufficiently deep inside the interior of Σ (this property is called *centering* in [2]), and at the same time the surface does not wiggle too much. Obviously we have $\lambda(\Sigma) \leq \max_{x \in \Sigma} |x|$. Using the terminology in [2], every admissible surface Σ satisfying (16) with $R, \delta^{-1} \gg 1$ is *on-center*

For the class of admissible surfaces, we have the following well-posedness result for (1):

Theorem 2 ([3, Thm. 5.3])**.** *Assume M is C^4-close to Schwarzschild and satisfies* (12)-(13)*. Then for $R \gg 1$, $\delta \ll 1$ and every admissible surface Σ_* satisfying* (16)*, there exists a global solution to* (1) *with initial configuration $\Sigma|_{t=0} = \Sigma_*$.*

Stationary solution to (1) are called *surfaces of Willmore type*. The existence and stability of such surfaces are studied in [2, 3].

Theorem 3 ([4, Thm. 1], [3, Thm. 5.3])**.** *Assume M is C^4-close to Schwarzschild and satisfies* (12)-(13)*. Then there exists a compact subset $K \subset M$, such that $M \setminus K$ is foliated by surfaces of Willmore type.*

Moreover, for $R \gg 1$, $\delta \ll 1$ and every admissible surface Σ_* satisfying (16)*, the flow generated by Σ_* under* (1) *converges smoothly to one of the leave of this foliation.*

References

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