Notes On the Stability of Cylindrical Singularities of the Mean Curvature Flow

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1 Introduction

Consider the mean curvature flow (MCF) for a family of hypersurfaces given by immersions $X(\cdot, t) : \mathbb{R}^{n-k} \times \mathbb{R}^{k+1} \to \mathbb{R}^{n+1}$, satisfying

$$\partial_t X = -H(X)\nu(X). \tag{1}$$

In this paper, we are interested in the dynamical behaviour of a solution X to (1), which first develops a singularity at $0 \in \mathbb{R}^{n+1}$, $t = T > 0$.

Equation (1) is invariant under rotation, translation, and parabolic rescaling. Motivated by these symmetries, we consider the following time-dependent rescaling for a solution to (1) as follows:

$$X(x, \omega, t) = \lambda(t)g(t)Y(y(x, t), \omega, \tau) + (0, z(t)), \tag{2}$$

where the immersion Y is defined through this relation, and

$$g(t) \in SO(n + 1), \quad g(0, x') = (0, x') \text{ for every } x' \in \mathbb{R}^{k+1}, \tag{3}$$

$$z(t) \in \mathbb{R}^{k+1}, \tag{4}$$

$$\lambda(t) := \left(2\int_t^T a(t')\, dt'\right)^{1/2}, \quad a(t) > 0, \tag{5}$$

$$y(x, t) := \lambda(t)^{-1}x, \tag{6}$$

$$\tau(t) := \int_0^t \lambda(t')^{-2}\, dt'. \tag{7}$$

Notice that we do not fix the parameter $\sigma := (g, z, a)$, but rather regard this as a path to be determined in the manifold

$$\Sigma := SO(n + 1) \times \mathbb{R}^{k+1} \times \mathbb{R}_{>0}. \tag{8}$$

The condition (5) shows that $\lambda(t)$ is uniquely determined by the path $a(t) > 0$. Indeed, this $\lambda(t)$ is the unique solution to the Cauchy problem

$$\dot\lambda(t)\lambda(t) = -a(t), \quad \lambda(T) = 0.$$

Here and in the remaining of this section, the dot denotes differentiation w.r.t. the fast time t-variable. By definition, the terminal condition on λ ensures the rescaling (2) gives rise to a tangent flow $Y = Y(y, \omega, \tau)$ in the microscopic variable y and slow time variable τ.

Direct computation shows X of the form (2) solves (1) if and only if the pair (σ, Y) solves

$$\partial_\tau Y = -H(Y)\nu(Y) - a\langle y, \nabla_y\rangle Y + aY - g^{-1}\partial_\tau gY - \lambda^{-1}g^{-1}\partial_\tau z. \tag{9}$$

To get (9), one uses the relations $\lambda\dot\lambda = -a$, $\lambda^2\dot y = ay$, and $\dot\tau = \lambda^{-2}$, which follow from (5)-(7) respectively. We call (9) *the rescaled mean curvature flow.*

The rescaled MCF (9) has the following family of stationary solutions:

$$Y_{a_0} \equiv \left(y, \sqrt{\frac{k}{a_0}}\omega\right), \tag{10}$$

$$\sigma_0 \equiv (g_0, z_0, a_0) \in \Sigma. \tag{11}$$

Geometrically, the three components in σ_0 consist of a rotation g_0 of the cylindrical axis, a transversal translation z_0, and a dilation by a factor of $\sqrt{k/a_0}$. The pair (σ, Y) corresponds to a cylinder with unit radius along the y axis, transformed by the symmetry σ_0 as in (2).

1.1 Main Results

In the remaining of this paper, we seek maximal solution X to (1) on the spatial domain $\mathbb{R}^{n-k} \times \mathbb{S}^k$ and the time interval $0 \le t < T$, of the form

$$X(x,\omega,t) = \lambda(t)g(t)\underbrace{\left(y(x,t), \left(\sqrt{\frac{k}{a(t)}} + \xi(y(x,t),\omega,\tau(t))\right)\omega\right)}_{\text{as } Y \text{ in (2)}} + (0, z(t)). \tag{12}$$

In terms of the blow-up variables y, τ from (6)-(7), this amounts to finding a pair (σ, Y) that solves the rescaled MCF (9) for all $\tau \ge 0$. Here $\sigma = \sigma(\tau)$ is a path of parameters. $Y = Y(\xi)$ is a flow of graphs over a fixed cylinder, parametrized by a path of functions $\xi(\cdot,\tau) : \mathbb{R}^{n-k} \times \mathbb{S}^k \to \mathbb{R}_{\ge 0}$, $\tau \ge 0$, as in (12).

With this convention as well as the rescaling (3)-(7) understood, the main result of this paper is the following assertions about the rescaled MCF (9):

Theorem 1. *Let $X^s(a)$, $s \ge 2$, $a > 0$ be the weighted Sobolev space defined in* (20). *There exists $0 < \delta \ll 1$ s.th. the following holds:*

1. *(Global existence) For every $a_0 \ge 1/2 + 2\delta$, there exists a subspace $\mathcal{S} \subset X^s(a_0)$ with finite codimensions, together with a map*

   $$\Phi : \mathcal{B}_\delta \cap \mathcal{S} \to X^s \equiv X^s(1/2), \quad \mathcal{B}_\delta, \mathcal{S} \text{ as in Definition 4,}$$

   satisfying

   $$\|\Phi(\eta_0)\|_s \lesssim \|\eta_0\|^2_{X^s}, \tag{13}$$
   $$\|\Phi(\eta_0) - \Phi(\eta_1)\|_{X^s} \lesssim \delta \|\eta_0 - \eta_1\|_{X^s}, \tag{14}$$

   for every $\eta_0, \eta_1 \in \mathcal{B}_\delta \cap \mathcal{S}$, as well as the following properties:

   For every $\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S}$, there exists a unique global solution (σ, Y) to the rescaled MCF (9) with initial configuration

   $$Y|_{\tau=0} = (y, (\sqrt{k/a_0} + \eta_0 + \Phi(\eta_0)\omega)$$

   on $\mathbb{R}^{n-k} \times \mathbb{S}^{n-k} \times \mathbb{R}_{\ge 0}$.

   Moreover, this solution Y is uniquely determined by the decomposition

   $$Y = (y, (\sqrt{k/a} + \xi(y,\omega,\tau))\omega) \quad (y \in \mathbb{R}^{n-k}, \omega \in \mathbb{S}^k), \tag{15}$$

   where $a = a(\tau)$ is a component of the path $\sigma(\tau) = (g(\tau), z(\tau), a(\tau)) \in \Sigma$, and $\xi = \xi(\cdot,\tau)$ is a path of functions on $\mathbb{R}^{n-k} \times \mathbb{S}^{n-k}$. The path (σ,ξ) lies in the space

   $$(\sigma,\xi) \in Lip([0,\infty),\Sigma) \times (C([0,\infty), X^s) \cap C^1([0,\infty), X^{s-2})).$$

2. *(Effective dynamics) Moreover, the path σ evolves according to the following system of ODEs:*

   $$\partial_\tau \sigma = \vec{F}(\sigma)\xi + \vec{M}(\sigma,\xi) \quad \text{for a.e. } \tau, \tag{16}$$
   $$\sigma(0) = (\delta_{ij}, 0, a_0) \in \Sigma, \tag{17}$$

   where the vector fields $\vec{F}$, $\vec{M}$ are as in (42).

3. *(Dissipative estimates) Moreover, the path $\sigma(\tau)$ dissipates to $\sigma(0)$, with the decay estimate*

   $$|\sigma(\tau) - \sigma(0)| \lesssim \delta \langle\tau\rangle^{-1}, \quad \tau \ge 0. \tag{18}$$

   Here and below, we write $\langle\cdot\rangle := (1 + |\cdot|^2)^{1/2}$.

   Moreover, the function ξ in the decomposition (15) *is non-negative for all τ, and dissipates to zero, with the decay estimate*

   $$\|\xi(\cdot,\tau)\|_{X^s} \le \delta \langle\tau\rangle^{-2}, \quad \tau \ge 0. \tag{19}$$

The result above are obtained by studying a quasilinear PDE (25) in ξ, coupled to a system of ODEs, (36)-(38) in the parameter σ. Up to a rigid motion and a dilation, initial configuration (17) can be replaced with any $\sigma_0 \in \Sigma$.

In [1, Thm. 4.31], Colding and Minicozzi have shown that the cylindrical singularity of the MCF is F-unstable. In terms of the PDE (25), this means that the static solution $\xi = 0$ is *linearly unstable.* In view of this, Theorem 1 above states that for a generic (i.e. finite codimensional) class of initial perturbations, $\xi = 0$ is *asymptotically stable* under the evolution (25). This gives a justification of the generality of cylindrical singularity based on the PDE ground.

2 The Modulation Equations: First-order Correction

In this section we develop the first order correction for the rescaled mean curvature flow.

By the relations (2) and (12), as far as Theorem 1 is concerned, it suffices to consider (9) in the unknown pair (σ, ξ), entering the equation through (12).

There are various advantage of studying (9) instead of (1). This is typical when one is interested in blow-up solutions to evolution equations, such as in the study of critical solitary wave dynamics where in general solution blows up with various profiles in finite time. See for instance [11]. For one, (9) has global solutions in time. (Actually, Theorem 3 below proves global well-posedness for (9) for a large class of initial configurations.) Thus we will mostly work with (9) and only return to (1) in Section 3.3, when we derive geometric consequences of the effective dynamics for the original flow. Moreover, since Theorem 1 only concerns with solutions that are normal graphs over $\mathbb{R}^{n-k} \times \mathbb{S}^k$ of the form (12), when we study (9) below, we reparametrize Y as $Y = (x, v(y, \omega, \tau)\omega)$ and analyze the behaviour of the path of functions $v(\cdot, \tau) : \mathbb{R}^{n-k} \times \mathbb{S}^k \to \mathbb{R}$, $\tau \geq 0$.

To this end, following [1, 2], we introduce a family of Gaussian weighted Sobolev spaces.

Definition 1. For $s \geq 0, a > 0$, define the space

$$X^s(a) := H^s(\mathbb{R}_y^{n-k} \times \mathbb{S}_\omega^k, \mathbb{R}; \rho_a), \quad \rho_a := e^{-a|y|^2/2}\, d\mu. \tag{20}$$

Here $d\mu$ is the canonical measure on $\mathbb{R}^{n-k} \times \mathbb{S}^k$.

The space $X^s(a)$ is equipped with the weighted inner product

$$\langle \phi, \psi \rangle_a = \int \phi\psi\rho_a \quad (\phi, \psi \in X^s(a)). \tag{21}$$

The induced norm by this inner product on $X^s(a)$ is denoted by $\|\cdot\|_{s,a}$. For simplicity, we write $X^s \equiv X^s(1/2)$ with norm $\|\cdot\|_s$. The space X^s has standard Gaussian measure $e^{-|y|^2/4} d\mu$ centered at $y = 0$. (The choice of a pivot $a = 1/2$ can be replaced by any other number.) For a linear map $L : X^s(a) \to X^r(b)$, we denote the operator norm of L as $\|L\|_{s,a\to r,b}$.

For $s \leq r$ and $0 < b \leq a$, there holds the trivial continuous embedding

$$X^r(b) \subset X^s(a). \tag{22}$$

In Appendix A, we consider the issue of estimating the weaker $X^s(b)$-norm in terms of the $X^s(a)$-norm, with $a \geq b$.

Following [12], we introduced the weighted volume, or the F-functional, for a normal graph $v = v(y, \omega)$ over cylinder as

$$F_a(v) := \int_S \rho_a\, d\mu_S, \quad S = S(v) := \left\{(y, v(y, \omega)\omega) : y \in \mathbb{R}^{n-k},\ \omega \in \mathbb{S}^k\right\}. \tag{23}$$

Using Sobolev inequalities, one can show that this functional is C^2 as a Fréchet-differentiable map on the space $X^s(a)$ with $a > 0$, $s \geq 2$. Indeed, denotes $F_a'(v)$ the $X^0(a)$-gradient of F_a at v. Explicitly, we have

$$F_a'(v) = -\Delta_y v + a\langle y, \nabla_y\rangle v - v^{-2}\Delta_\omega v - av + kv^{-1} + N_1(v), \tag{24}$$

where $N_1(v) : X^s(a) \to X^{s-2}(a)$ is a quasilinear elliptic operator, given explicitly in (**??**). Expression (24) can be derived from the first variation formula in [1], and the graphical MCF equation obtained in [3]. The nonlinear map F_a' is C^1 from $X^s(a) \to X^{s-2}(a)$.

2.1 The graphical equations

In this subsection we consider (9) in the space $X^s(a)$ with $a > 0$, $s \geq 2$.

- In the remaining of this paper, unless otherwise stated, the dot denotes $\frac{\partial}{\partial\tau}$.

Lemma 1. *X of the form (12) satisfies the MCF (1) if and only if the pair (σ, ξ) satisfies*

$$\dot{\xi} = -F_a'(\sqrt{k/a} + \xi) - \partial_\sigma W(\sigma)\dot{\sigma}, \tag{25}$$

where $F_a'(v)$ is as in (24), and $W : \Sigma \to X^s(a)$ is given by

$$W(\sigma) := \sqrt{k/a} + g_{n-k+l,j}\omega^l y^j + \left\langle z,\, \lambda^{-1}\omega\right\rangle, \tag{26}$$

Proof. Write $v = \sqrt{k/a} + \xi$. Recall from the Introduction that X of the form (12) satisfies the MCF (1) if and only if the rescaled hyersurface $Y = (y, v(y, \omega)\omega, \tau)$ from (2) solves (9). Following [5–8, 10, 16], we derive the evolution of v from (9).

Taking inner product on both sides of (9) with the vector $(0, \omega) \in \mathbb{R}^{n+1}$, we find

$$\dot{v} = -\langle H(v)\nu(v), (0, \omega)\rangle - a\langle y, \nabla_y\rangle v + av - \langle \dot{g}(y, v\omega), (0, \omega)\rangle - \lambda^{-1}\langle \dot{z}, \omega\rangle. \tag{27}$$

Here $H(v)$ and $\nu(v)$ denotes the mean curvature and the unit normal vector at the point $(y, v(y, \omega)\omega)$ respectively. To get (27), we use the fact that $\langle g^{-1}(\cdot), \cdot\rangle = \langle \cdot, g(\cdot)\rangle$ as $g \in SO(n+1)$, and the requirement $g(0, x') = (0, x')$ for every $x' \in \mathbb{R}^{k+1}$.

The first term in the r.h.s. is calculated e.g. in [8, Appendix A], whose derivation is routine. This, together with the following two terms, constitute the r.h.s. of (24)

Differentiating the condition (3), we find $\dot{g}(0, x') = 0$. Thus the penultimate term in the r.h.s. of (27) further simplifies as

$$\langle \dot{g}(y, v\omega), (0, \omega)\rangle = \dot{g}_{n-k+l,j}\omega^l y^j \quad (1 \leq j \leq n-k,\ 1 \leq l \leq k+1).$$

Lastly, moving the τ-derivative of $\sqrt{k/a}$ in the l.h.s. of (27) (recall $v(\tau) = \sqrt{k/a(\tau)} + \xi(\tau)$), gives (25). □

2.2 Linearized operator at Y_a

The only nonlinear term in (25) is contained in the map F'_a, which is the (normal) $X^0(a)$-gradient of the F-functional, (23). In this subsection we linearize this map around the stationary solution $v \equiv \sqrt{k/a}$, which corresponds to a cylinder with radius $\sqrt{k/a}$ and along the y axis. Then we study the zero-unstable modes of the linearized operator.

All of the results in this section are known by far. See for instance [1, Sec.5], [2, Sec. 3.2].

Lemma 2. *The linearized operator of $F'_a(v)$ at the critical point $v \equiv \sqrt{k/a}$ is given by*

$$L(a) := -\mathcal{L}_a - \frac{a}{k}\Delta_\omega - 2a, \tag{28}$$

where $\mathcal{L}_a := \Delta_y - a\langle y, \nabla_y(\cdot)\rangle$ is the drift Laplacian on the weighted space $X^s(a)$.

Moreover, the operator $L(a)$ is self-adjoint in $X^s(a)$ w.r.t. the inner product from (21)*, and is bounded from $X^s(a) \to X^{s-2}(a)$. The spectrum of $L(a)$ is purely discrete, and the only non-positive eigenvalues, together with the associated eigenfuncitons, are given by*

$$-2a, \quad \text{with eigenfunction } \Sigma^{(0,0)(0,0,0)}(a) := -\frac{\sqrt{k}}{2}a^{-3/2}, \tag{29}$$

$$-a, \quad \text{with eigenfunctions } \Sigma^{(0,1)(0,0,l)}(a) := \lambda^{-1}\omega, \tag{30}$$

$$-a, \quad \text{with eigenfunctions } \Sigma^{(1,0)(i,0,0)}(a) := \frac{1}{\|y^i\|^2_{0,a}}y^i, \tag{31}$$

$$0, \quad \text{with eigenfunctions } \Sigma^{(1,1)(i,0,l)}(a) := y^i\omega^l, \tag{32}$$

$$0, \quad \text{with eigenfunctions } \Sigma^{(2,0)(i,j,0)}(a) := \frac{1}{\|ay^iy^j - \delta_{ij}\|^2_{0,a}}(ay^iy^j - \delta_{ij}). \tag{33}$$

Here and in the remaining of this paper, the indices $l = 1, \dots, k+1$ and $i, j = 1, \dots, n-k$.

Moreover, the functions in (29)*-*(32) *are mutually orthogonal w.r.t. the inner product* (21)*.*

Remark 1. Notice that we do not normalize (29),(30) and (32). The point is that these correspond to various components of the Fréchet derivative $\partial_\sigma W(\sigma)$, where W is as in (26).

Thus, with L_a given in (28), we can rewrite (25) as

$$\dot{\xi} = -L(a)\xi - N(a,\xi) - \partial_\sigma W(\sigma)\dot{\sigma}, \tag{34}$$

where the nonlinearity $N(a,\xi)$ is defined by this expansion. This map is calculated explicitly in (**??**). In Section **??**, we study some key properties of the nonlinearity.

2.3 The modulation equation

In this subsection we study (34), which is equivalent to (25) and the rescaled MCF, (9). Equation (34) is the central subject in the remaining sections.

It is important to note that *some, but not all of* the zero-unstable modes of the linearized operator $L(a)$ are due to broken symmetries. Indeed, for $\Sigma^{(m,n)}(a)$ with $(m,n) = (0,0),(0,1),(1,1)$, there exists a path $\sigma(s) \in \Sigma$ (the manifold defined in (8)) s.th.

$$\Sigma^{(m,n)(i,j,l)}(a) = \partial_s|_{s=0}T_{\sigma(s)}Y_a.$$

Here $T_\sigma = T_{g,z,a}$ denotes the action of symmetry on the graph function, derived from (2), and Y_a is the cylinder from (10). For example, take $\sigma(s) = (g_0, z(s), a_0)$ with a path $z(s) \in \mathbb{R}^{k+1}$, $\partial_s|_{s=0}z(s) = \lambda^{-1}e^1$. Then

$$\partial_s|_{s=0}T_{\sigma(s)}Y_{a_0} = \partial_s|_{s=0}(\sqrt{k/a_0} + z_l(s)\omega^l) = \Sigma^{(0,1),(0,0,1)}(a_0).$$

Consequently, with a proper choice of σ as a function of ξ, we can eliminate these modes. This is the content of the next lemma.

Lemma 3. *Suppose the pair (σ,ξ) is a global solution to* (25)*, s.th.*

$$\left\langle \xi(0),\, \Sigma^{(m,n)}(a(0)) \right\rangle_{a(0)} = 0$$

for $(m,n) = (0,0),(0,1),(1,1)$. Then ξ satisfies the orthogonality condition

$$\left\langle \xi(\tau),\, \Sigma^{(m,n)}(a(\tau)) \right\rangle_{a(\tau)}, \quad \tau \geq 0,\ (m,n) = (0,0),(0,1),(1,1), \tag{35}$$

if and only if $\sigma = (g, z, a)$ *satisfies the modulation equations:*

$$\left\|\Sigma^{(1,1)(j,0,l)}(a)\right\|_{0,a}^{2} \dot{g}_{n-k+l,j} = - \left\langle N(a,\xi),\, \Sigma^{(1,1)(j,0,l)}(a)\right\rangle_a , \tag{36}$$

$$\begin{aligned}\left\|\Sigma^{(0,1)(0,0,l)}(a)\right\|_{0,a}^{2} \dot{z}_l =& a \left\langle \xi,\, \Sigma^{(0,1)(0,0,l)}(a)\right\rangle_a \\ &- \left\langle N(a,\xi),\, \Sigma^{(0,1)(0,0,l)}(a)\right\rangle_a \\ &+ \left\langle \xi,\, \partial_\tau \Sigma^{(0,1)(0,0,l)}(a)\right\rangle_a ,\end{aligned} \tag{37}$$

$$\begin{aligned}\left\|\Sigma^{(0,0)}(a)\right\|_{0,a}^{2} \dot{a} =& 2a \left\langle \xi,\, \Sigma^{(0,0)}(a)\right\rangle_a \\ &- \left\langle N(a,\xi),\, \Sigma^{(0,0)}(a)\right\rangle_a + \left\langle \xi,\, \partial_\tau \Sigma^{(0,0)}(a)\right\rangle_a .\end{aligned} \tag{38}$$

Remark 2. The terminology *modulation equations* is common in the study of solitary wave dynamics, e.g. [4,9,14,15], and refers to equations of the form (36)-(38), derived from a condition as (35). We adopt the same terminology here.

Proof. Differentiating both sides of (35) w.r.t. τ, we find that (35) holds for all $\tau \geq 0$ if and only if

1. (35) holds for $\tau = 0$;
2. For $\tau > 0$, there holds
$$\left\langle \dot{\xi}(\tau),\, \Sigma^{(m,n)}(a(\tau))\right\rangle_{a(\tau)} = - \left\langle \xi(\tau),\, \partial_\tau \Sigma^{(m,n)}(a(\tau))\right\rangle_{a(\tau)} . \tag{39}$$

The first point is in the assumption. For the second point, at time τ, taking the $a(\tau)$-weighted inner product (see (21)) of both sides of (34), using the spectral property of $L(a)$ shown in Lemma 2, together with the orthogonality among $\Sigma^{(m,n)}$ as $L(a)$ is self-adjoint in $X^s(a)$, we conclude that (39) is equivalent to (38). □

From (36)-(38) we see that if we have a priori control of ξ in X^s-norm, then we can ensure that the path $\sigma(\tau)$ stays approximately constant. This is the usual low velocity condition for an adiabatic theory.

3 The Quadratic Correction Φ and Its Properties

In this section we define the key map Φ in Theorem 1, which originates from a fixed point scheme. Then we prove Theorem 1, assuming the results in Sects. 4-6. These results are inspired by [13] and we use some of the notations there.

From now on we will often work with $X^s(a)$ spaces with different a. In order to facilitate this effort, we introduce the following notion of admissible paths:

Definition 2. Fix $0 < \delta \ll 1$. A path $\sigma(\tau) = (g(\tau), z(\tau), a(\tau)) \in \Sigma$ is admissible if the following holds:

1. The map $\tau \mapsto \sigma(\tau)$ is continuous and locally Lipschitz.
2. The entire path $a(\tau)$ lies in the δ-neighbourhood of $a(0)$, i.e.
$$\sup_{\tau\geq 0} |a(\tau) - a(0)| \leq \delta.$$
3. Initially, $a(0) \geq 2\delta + 1/2$;
4. The path $z(\tau) < \sqrt{k/a} e^{\int^\tau a}$, or equivalently $z(\tau)/\lambda(\tau) < \sqrt{k/a}$ for all τ.

Item 1 above is a standard regularity requirement to make sense of the velocity of the path σ. Items 2-3, which are the most important ones, are imposed for several reasons. First, we want to study (25) in the space $X^s \equiv X^s(1/2)$, while from time to time we consider the stronger norm $\|\cdot\|_{s,a(\tau)}$. These conditions ensure $a(\tau) \geq 1/2$ and therefore $X^s \subset X^s(a(\tau))$ for all τ by (22). Secondly, for some estimates, e.g. to bound a projection into an eigenspace of $L(a)$, we need a to stay within a fixed range, so that these estimates can be made independent of a. Thirdly, for some other estimates, e.g. the nonlinear estimate (**??**), we need $a(\tau)$ to stay a fixed distance away from the pivotal number 1/2. Item 4 above is imposed so that the function W in (26) remains bounded.

3.1 The definition of Ψ

In this subsection, we introduce another key map Ψ. This map goes into the definition of the map Φ in Theorem 1.

Consider the following linear equation obtained by freezing coefficients in (34) and the modulation equations (36)-(38) at a fixed path

$$(\sigma^{(0)}, \xi^{(0)}) \in Lip([0,\infty), \Sigma) \times (C([0,\infty), X^s) \cap C^1([0,\infty), X^{s-2})), \tag{40}$$

given as follows:

$$\dot{\xi} = -L(a^{(0)})\xi - N(a^{(0)}, \xi^{(0)}) - \partial_\sigma W(\sigma^{(0)})\dot{\sigma}, \tag{41}$$

$$\dot{\sigma} = \vec{F}(\sigma^{(0)})\xi + \vec{M}(\sigma^{(0)}, \xi^{(0)}), \tag{42}$$

where $\vec{F}$, $\vec{M}$ are defined by isolating the linear and nonlinear terms in ξ respectively in (36)-(38). For instance, the last entry of $\vec{F}(\sigma^{(0)})\xi$ is given by

$$-\frac{1}{\left\|\Sigma^{(0,0)}(a^{(0)})\right\|^2_{a^{(0)}}}\left(2a^{(0)}\left\langle \xi,\, \Sigma^{(0,0)}(a^{(0)})\right\rangle_{a^{(0)}} + \left\langle \xi,\, \partial_\tau\Sigma^{(0,0)}(a^{(0)})\right\rangle_{a^{(0)}}\right),$$

and the last entry of $\vec{M}(\sigma^{(0)}, \xi^{(0)})$ is given by

$$-\frac{1}{\left\|\Sigma^{(0,0)}(a^{(0)})\right\|^2_{a^{(0)}}}\left\langle N(a^{(0)}, \xi^{(0)}),\, \Sigma^{(0,0)}(a^{(0)})\right\rangle_{a^{(0)}}.$$

To (41)-(42) we associate the initial configuration

$$\sigma(0) = (\delta_{ij}, 0, a_0) \quad \text{for some fixed } a_0 > 1/2, \tag{43}$$

$$\xi(0) = \eta_0 + \beta_i \Sigma^{(1,0)(i,0,0)}(a_0) + \gamma_{ij}\Sigma^{(2,0),(i,j,0)}(a_0), \tag{44}$$

where $\eta_0 \in X^s(a_0)$ is fixed, and and $\beta_i, \gamma_{ij} \in \mathbb{R}$ are to be chosen later as functions of η_0.

In the remaining of this paper, the central object is the Cauchy problem (41)-(44). In Appendix **??**, we show the linear system (41)-(42) is equivalent to a single linear equation

$$\dot{\xi} + L(a^{(0)})\xi = -\mathcal{W}(\sigma^{(0)})\xi - \tilde{N}(a^{(0)}, \xi^{(0)}), \tag{45}$$

where the maps $\mathcal{W}$, $\tilde{N}$ are defined in (**??**). We also show the Cauchy problem (41)-(44) has a unique global solution in the space (40).

Now, we set up a contraction scheme in the following space:

Definition 3. Fix $0 < \delta \ll 1$. The space $\mathcal{A}_\delta = \mathcal{A}^\sigma_\delta \times \mathcal{A}^\xi_\delta$ consists of

$$(\sigma, \xi) \in Lip([0,\infty), \Sigma) \times (C([0,\infty), X^s) \cap C^1([0,\infty), X^{s-2})),$$

s.th. the following holds:

1. $\sigma(0)$ is as in (43), with $a_0 \geq \frac{1}{2} + 2\delta$;
2. For some fixed $c_0 > 0$, there hold the decay estimates

$$|\dot{\sigma}(\tau)| \leq c_0\delta\left\langle\tau\right\rangle^{-2}, \quad \tau \geq 0, \tag{46}$$

$$\left\|\xi(\tau)\right\|_s \leq \delta\left\langle\tau\right\rangle^{-2}, \quad \tau \geq 0; \tag{47}$$

3. Let $b := \frac{1}{2} - 4\delta$. There holds the pivot condition from Lemma 4, i.e.

$$\left\|\xi(\tau)\right\|^2_{s,b} \leq c, \quad \tau \geq 0, \tag{48}$$

for some fixed $c > 0$.

Clearly, $\mathcal{A}_\delta$ is not empty, as the pair $(\sigma, \xi) \equiv (\sigma(0), 0)$ lies in this space.

Item 1 above amounts to fixing an initial cylindrical coordinate. Up to a rigid motion in $\mathbb{R}^{n+1}$ and an initial dilation, (43) can be replaced by any other parameters. The decay conditions in Item 2, (46)-(47), are the most important ones here, for the following reasons. First, these correspond to the claimed decay properties from Theorem 1, and finding a fixed point in $\mathcal{A}_\delta$ amounts to constructing a global solution to (45). Secondly, if $\sigma \in \mathcal{A}^\sigma_\delta$, then σ is admissible as in Definition 2, and we have the convenient properties mentioned earlier. Item 4 has to do with the interpolation from Lemma 4, whose importance is explained in Section A. Of course, if (48) holds, then $\xi(\tau) \in X^s(b)$. But we never use this fact other than a pivot condition for deriving estimates in X^s.

Consider the solution map

$$\Psi : (\sigma^{(0)}, \xi^{(0)}) \mapsto \text{ the unique solution } (\sigma, \xi) \text{ to (41)-(44)}. \tag{49}$$

In Lemma **??**, we show this map is well-defined. Hereafter we want to show that

1. $\Psi(\mathcal{A}_\delta) \subset \mathcal{A}_\delta$;
2. $\Psi : \mathcal{A}_\delta \to \mathcal{A}_\delta$ is a contraction w.r.t. a suitable norm on $\mathcal{A}_\delta$.

In Section 4, we show the first point holds for appropriately chosen initial configurations (44). In Section 5 we show the second point holds if $\delta \ll 1$.

Remark 3. Consider the initial configuration (44). The constants β_i and γ_{ij} are to be determined later as a function of $\eta_0 \in X^s(a_0)$. This way the map Ψ depends only on the function η_0, i.e. $\Psi = \Psi(\cdot, \eta_0)$.

Indeed, if $\eta_0 = 0$, then it is easy to see that the fixed point of $\Psi(\cdot, 0)$ is just the vector $(\sigma, \xi) \equiv (\sigma(0), 0)$ in $\mathcal{A}_\delta$. This corresponds to the trivial static solution of (34), namely the cylinder of radius $\sqrt{k/a_0}$. On the other hand, if $\eta_0 \neq 0$, then in general $\Psi(0, \eta_0) \neq 0$ by (49).

For simplicity of notation, most of the time we do not write $\Psi = \Psi(\cdot, \eta_0)$ but let this be understood.

3.2 The definition of Φ

For a function η, the map Φ is defined in terms of the fixed point of $\Psi = \Psi(\cdot, \eta)$ (see Remark 3). But first, we need to introduce a suitable parameter space for Φ.

Definition 4. Let $a_0 > 1/2$ be as in (43). Let $0 < \delta \ll 1$, $0 < b < 1/2$ be as in Definition 3. Let $C > 0$ be a large constant depending on the absolute, implicit constant in (63).

The space

$$\mathcal{B}_\delta \subset X^s(b) \subset X^s \subset X^s(a_0)$$

consists of all functions $\eta = \eta(y, \omega)$ s.th.

$$\|\eta\|_s < \delta, \tag{50}$$

$$\|\eta\|_{s,b}^2 \leq c/2, \quad \text{where the number } c > 0 \text{ is as in (48)}, \tag{51}$$

$$\eta(y,\omega) \geq \delta \text{ and } |\eta(y,\omega)| \geq C\delta^2 |y|^2 \text{ for } |y|^2 \geq C^{-1}\delta^{-1}. \tag{52}$$

The space $\mathcal{S} = X^s(a_0)$ is the orthogonal complement to all the zero-unstable modes in (29)-(33) of the linearized operator $L(a_0)$ w.r.t. the $X^0(a_0)$-inner product defined in (21).

We will explain the the meaning of this parameter space below. For now, observe that the codimension of $\mathcal{S}$ is the sum of the multipliciteis of all the non-positive eigenvalues of $L(a_0)$, which equals to

$$\operatorname{codim} \mathcal{S} = n + 2 + \frac{(n-k)(n-k+3)}{2}.$$

Notice that, as pointed out in [2, Eqn. (3.44)], not all of the zero-unstable modes in (29)-(33) are linearly independent.

Now we define the key map Φ in Theorem 1.

Definition 5 (the quadratic correction Φ)**.** For $\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S}$ as in Definition 4, define

$$\Phi(\eta_0) := \beta_i(\eta_0)\Sigma^{(1,0)(i,0,0)}(a_0) + \gamma_{ij}(\eta_0)\Sigma^{(2,0),(i,j,0)}(a_0), \tag{53}$$

where β_i, γ_{ij} are defined in Theorem 3.

Remark 4. Consider the requirements from Definition 4. In view of the quadratic estimates (13), for sufficiently small δ, conditions (50)-(51) ensure the compatibility to impose the initial condition $\xi(0) = \eta_0 + \Phi(\eta_0)$ for a path $\xi \in \mathcal{A}_\delta^\xi$. (I.e. this ensures $\|\xi(0)\|_s \leq \delta$ for sufficiently small δ). Condition (52) has to to with the geometric interpretation of Theorem 1. Indeed, for sufficiently small δ, by the definition (53) above and the formulae (31), (33), condition (52) ensures the function $\eta_0 + \Phi(\eta_0) \geq 0$ on the entire cylinder. In this case, we can interpret this function as a normal graph over the the cylinder, parametrizing the hypersurface

$$Y_0 = (y, (\sqrt{k/a_0} + \eta_0 + \Phi(\eta_0))\omega), \quad y \in \mathbb{R}^{n-k},\ \omega \in \mathbb{S}^k.$$

By the avoidance principle for MCF, it follows that a flow satisfying (9) with initial configuration Y_0 above remains to be a normal graph over the cylinder of radius $\sqrt{k/a_0}$ for all time. This justifies the geometric meaning for the analysis of PDEs in terms of the graph function ξ in the remaining sections.

Using the modulation equations from Section 2.3, we can ensure that a flow generated by a function $\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S}$ under (34) remains orthogonal at all $\tau \geq 0$ to all of the zero-unstable modes that are due to broken symmetry (see a discussion at the beginning of Section 2.3). Yet, this does not suffice to give any dissipative estimate because of the remaining zero-unstable modes that cannot be eliminated by the modulation equations, namely $\Sigma^{(1,0)}$ (horizontal translation) and $\Sigma^{(2,0)}$ (shape instability). The former is protected by the symmetry of cylinder lying along the y-axis. The latter is not due to any symmetry.

3.3 Proof of the Main Results

In this subsection we assume Theorem 2-Theorem 4 hold. Then we prove the main results from Section 1.

Proof of Theorem 1, assuming Theorem 2-Theorem 4. By construction, the fixed point (σ, ξ) in $\mathcal{A}_\delta$ of the map Ψ solves the graphical rescaled MCF (34), coupled to the modulation equations (36)-(38), with initial configuration given by

$$\sigma(0) = (\delta_{ij}, 0, a_0), \quad \xi(0) = \eta_0 + \Phi(\eta_0).$$

This amounts to a global solution to the rescaled MCF (9) satisfying

$$\dot{\sigma} = \vec{F}(\sigma)\xi + \vec{M}(\sigma, \xi), \quad Y(t) = (y, (\sqrt{k/a(t)} + \xi(t))\omega), \tag{54}$$

together with the initial configuration

$$\sigma(0) = (\delta_{ij}, 0, a_0), \quad Y(0) = (y, (\sqrt{k/a_0} + \eta_0 + \Phi(\eta_0))\omega). \tag{55}$$

Estimates (13)-(14) are the content of Theorem 4. Thus Parts 1-2 of the theorem is proved. The positivity of ξ follows from the fact that $\xi(0) \geq 0$, which holds by (52), and the avoidance principle for MCF, c.f. Remark 4. The remainder estimates (18)-(19) come from the decay condition in the space $\mathcal{A}_\delta$ in which Ψ is a contraction, namely (46)-(47).

$\square$

Proof of Corollary **??**. Without loss of generality, suppose $t_0 = 0$. Let X be a maximal solution to (1), and suppose for some $\lambda_0 > 0$, the hypersurface $\lambda_0^{-1} X(\cdot, 0) \in M$ with M given in (**??**). Then, by Theorem 1, X is of the form (12) for all $t < T$, and X is uniquely determined by the pair $\sigma(\tau(t)), \xi(y(x,t), \omega, \tau(t))$ constructed in Theorem 3 below.

In terms of the slow time variable τ, the Cauchy problem for σ in (16)-(17) uniquely determines the axis and radius of the limit cylinder given the initial profile. This system of ODEs does not depend on taking any particular sequence of $\lambda \to \infty$ in the rescaling procedure, and the solution to the Cauchy problem is unique by Theorem 3. Thus Corollary **??** follows. $\square$

4 The Mapping Property of Ψ

In this section we prove that for appropriately chosen constants β_i, γ_{ij} in (44) the Ψ maps from $\mathcal{A}_\delta$ into itself.

Recall that Ψ maps a fixed path $(\sigma^{(0)}, \xi^{(0)}) \in \mathcal{A}_\delta$ to the global solution to (45) with initial configuration (43)-(44), constructed explicitly in Lemma **??**.

Theorem 2. *For every $\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S}$ as in Definition 4 and every fixed path $(\sigma^{(0)}, \xi^{(0)}) \in \mathcal{A}_\delta$, there exist unique coefficients β_i, γ_{ij}, depending on the choice of $\sigma^{(0)}$, $\xi^{(0)}$ only, s.th. the solution to* (41)-(44) *lies in $\mathcal{A}_\delta$.*

Moreover, there hold the quadratic estimates

$$\left|\beta_i(\sigma^{(0)}, \xi^{(0)})\right| \lesssim \delta^2, \tag{56}$$

$$\left|\gamma_{ij}(\sigma^{(0)}, \xi^{(0)})\right| \lesssim \delta^2. \tag{57}$$

Moreover, for two given paths $\sigma^{(m)}$, $\xi^{(m)}, m = 1, 2$, there hold the Lipschitz estimates

$$\left|\beta_i(\sigma^{(0)}, \xi^{(0)}) - \beta_i(\sigma^{(1)}, \xi^{(1)})\right| \lesssim \delta \left\|(\sigma^{(0)}, \xi^{(0)}) - (\sigma^{(1)}, \xi^{(1)})\right\|, \tag{58}$$

$$\left|\gamma_{ij}(\sigma^{(0)}, \xi^{(0)}) - \gamma_{ij}(\sigma^{(1)}, \xi^{(1)})\right| \lesssim \delta \left\|(\sigma^{(0)}, \xi^{(0)}) - (\sigma^{(1)}, \xi^{(1)})\right\|, \tag{59}$$

$$\tag{60}$$

where the norm in the r.h.s. is defined in (61).

5 The Contraction Property of Ψ

In this section we prove the map $\Psi : \mathcal{A}_\delta \to \mathcal{A}_\delta$ defines a contraction w.r.t. the following norm:

$$\|(\sigma, \xi)\| = \sup_{\tau \geq 0}(c_0^{-1} \langle\tau\rangle |\sigma(\tau) - \tau(0)| + \langle\tau\rangle^2 \|\xi(\tau)\|_s). \tag{61}$$

Here $c_0 > 0$ is the constant in (46). This defines a norm equivalent to the uniform one in the space $C([0, \infty), \Sigma \times X^s)$, in which the the space $\mathcal{A}_\delta$ from Definition 3 is the closed ball of size $O(\delta)$ around zero w.r.t. the norm (61). This shows $(\mathcal{A}_\delta, \|\cdot\|)$ is complete.

Now we show Ψ is a contraction in $(\mathcal{A}_\delta, \|\cdot\|)$. This essentially implies all of the statements in Theorem 1.

Theorem 3. *The map $\Psi : \mathcal{A}_\delta \to \mathcal{A}_\delta$ defined in* (49) *is a contraction with respect to the norm* (61)*, satisfying*

$$\left\|\Psi(U^1) - \Psi(U^2)\right\| \leq \delta^{1/2} \left\|U^1 - U^0\right\| \quad (U^1, U^2 \in \mathcal{A}_\delta). \tag{62}$$

Consequently, there exist unique constants

$$\beta_i(\eta_0), \gamma_{ij}(\eta_0) = O(\delta^2), \tag{63}$$

s.th. there exists a unique solution to the nonlinear system (34), (36)-(38) *with initial condition* (43) *and this choice of β_i, γ_{ij} in* (44).

Remark 5. As a by-product, this proves a global well-posedness result for the rescaled MCF (9). The admissible set of initial configurations forms a finite-codimensional manifold in X^s. The global solutions constructed here does not necessarily arise from rescaling a given solution to (1).

6 A Sable Manifold Theorem for the Rescaled MCF

In this section we consider Theorem 1 as a stable manifold theorem for the rescaled MCF, (34).

Define a set

$$M := \{\eta + \Phi(\eta) : \eta \in \mathcal{B}_\delta \cap \mathcal{S}\} \quad (0 < \delta \ll 1). \tag{64}$$

This set corresponds to the one in (**??**) in the obvious way, and therefore is denoted with the same symbol. Recall that the map Φ, given in Definition 5, maps each $\eta \in \mathcal{B}_\delta \cap \mathcal{S}$ in the parameter space to the fixed point of the map $\Psi(\cdot, \eta)$ in the space $\mathcal{A}_\delta$.

In what follows, we show the effect of Φ amounts to a second order perturbation of $\mathcal{B}_\delta \cap \mathcal{S}$, and consequently the manifold M is non-degenerate. We conjecture that in fact Φ is a C^1 immersion.

We also show a Lipschitz estimate for Φ, This shows M is a Lipschitz graph over $\mathcal{B}_\delta \cap \mathcal{S}$. By Rademacher's theorem, this justifies the treatment of M as a manifold.

Theorem 4. *The map $\Phi = \Phi$ in Definition 5 satisfies*

$$\|\Phi(\eta_0)\|_s \lesssim \|\eta_0\|_s^2 \quad (\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S}), \tag{65}$$

$$\|\Phi(\eta_0) - \Phi(\eta_1)\| \lesssim \delta \|\eta_0 - \eta_1\| \quad (\eta_0, \eta_1 \in \mathcal{B}_\delta \cap \mathcal{S}). \tag{66}$$

A Elementary Lemmas

In this section we prove some abstract and elementary lemmas. Despite their simple appearcance, they play crucial roles in Sects. 4-5.

Recall that the weighted Sobolev space $X^s(a)$ is defined in Section 2. As discussed in that section, for two numbers $b \le 1/2 \le a$ we have the trivial embedding

$$X^s(b) \le X^s \equiv X^s(1/2) \subset X^s(a) \quad (s \ge 0).$$

In Sect. 4, we are concerned with the X^s-estimate for a family of functions $\phi(t) \in X^s(b)$, knowing some estimate of the varying norm $\|\phi(t)\|_{s,a(t)}$ for a function $a(t) \ge 1/2$. Lemma 4 below allows us to deal with this issue.

Definition 6 (pivot condition)**.** Fix some $0 < \delta < 1/8$. A family of functions $\phi(\cdot, t) : \mathbb{R}^{n-k} \times \mathbb{S}^k \to \mathbb{R}$, $t \ge 0$ satisfied *the pivot condition* if there is constant $c > 0$ independent of t s.th.

$$\int \sum_{|\alpha| \le s} |\partial^\alpha \phi(t)|^2 \rho_b \le c, \quad b := \frac{1}{2} - 4\delta > 0. \tag{67}$$

Equivalently, this means $\phi(t) \in X^s(b)$ and is uniformly bounded for all t in the $X^s(b)$-norm.

Lemma 4. *Let $a(t)$ be a function satisfying $0 \le a(t) - 1/2 \le 2\delta$. Suppose $\phi(t)$ satisfies the pivot condition* (67)*, and $\|\phi(t)\|_{s,a(t)} \le g(t)$ for some positive function g. Then there holds*

$$\|\phi(t)\|_s \le c^{1/12} g(t)^{2/3}. \tag{68}$$

Proof. Let $u := \sum_{|\alpha| \le s} |\partial^\alpha \phi|^2$. Then $u\rho_{1/2} = (u^{2/3}\rho_{2a/3})(u^{1/3}\rho_{\frac{1}{2}-\frac{2a}{3}})$. By Hölder's inequality with $p = 3/2$ and $q = 3$, we find

$$\|\phi\|_s^2 = \int u\rho_{1/2} \le \left(\int u\rho_a\right)^{2/3} \left(\int u\rho_{\frac{3}{2}-2a}\right)^{1/3} = \|\phi\|_{s,a}^{4/3} \left(\int u\rho_{\frac{3}{2}-2a}\right)^{1/3}. \tag{69}$$

The assumption $0 \le a - 1/2 \le 2\delta < 1/4$ implies $\frac{3}{2} - 2a \ge b > 0$, and therefore the second factor in the r.h.s. of (69) is bounded by the l.h.s. of (67). Hence (68) follows from the assumption $\|\phi\|_{s,a} \le g$ and taking square root on both sides of (69). □

Remark 6. More generally, for all $1 < p < \infty$, one can vary the pivot condition to obtain similar estimate as (68) with higher power in the r.h.s.. For our need, any power $1 < p < 2$ will do.

By Lemma 4, every dissipative estimate in terms of the $X^s(a)$-norm with possibly changing $a \ge 1/2$ gives a (weaker) dissipative estimate in the fixed X^s-norm. This is important for many estimates in Sect.4.

The following two lemmas are the mechanism behind Theorem 2 and Theorem 3, respectively. These results are obtained in [13].

The first lemma concerns with Cauchy problem for an inhomogeneous ODE.

Lemma 5 (c.f. [13, Lem. 23])**.** *Fix two functions $a(t) \ge c > 0$ and $f \in L^1([0,\infty), \mathbb{R})$. Consider the Cauchy problem for $x : [0, \infty) \to \mathbb{R}$:*

$$\dot{x} - a(t)x = f, \tag{70}$$

$$x(0) = x_0 \in \mathbb{R}. \tag{71}$$

There exists a unique bounded solution if and only if

$$x_0 = -\int_0^\infty f_i(t') e^{-\int_0^{t'} a}\, dt' \text{ in } (71). \tag{72}$$

Moreover, if (72) *holds, then the solution to* (70)-(71) *is given by*

$$x(t) = -\int_t^\infty f(t') e^{-\int_t^{t'} a},\, d\tau'. \tag{73}$$

Proof. The variation of parameter formula gives the general solution to (70)-(71) as

$$x(t) = e^{\int_0^t a}(x_0 + \int_0^t f e^{-\int_0^{t'} a}\, dt'). \tag{74}$$

Multiplying both sides of (74) by $e^{-\int_0^t a}$, and then taking $t \to \infty$, we find that if $x(t) \le C$, then $x(t)e^{-\int_0^t a} \to 0$ as $t \to \infty$ because of the uniform bound $a(t) \ge c > 0$, and therefore (72) holds.

Conversely, if (72) holds, then the general formula (74) simplifies to (73). For $a(t) > 0$ and $f \in L^1$ we can conclude from here that $x(t) \le \|f\|_{L^1}$.

□

For the next lemma, we adopt similar notations as in Sects. 4-6.

Lemma 6 (c.f. [13, Lem. 28])**.** *Let $A \subset X$ be a closed subset of a Banach space $(X, \|\cdot\|)$. Let $B \subset Y$ be a subset of a normed vector space $(Y, \|\cdot\|_Y)$. Let $\Psi : X \times Y \to X$ be a map s.th.*

1. *$\Psi(A \times B) \subset A$;*
2. *There exists $0 < \rho < 1$ s.th.*
$$\sup_{\eta \in B} \left\| \Psi(u^{(1)}, \eta) - \Psi(u^{(0)}, \eta) \right\| \leq \rho \left\| u^{(1)} - u^{(0)} \right\|$$
for two vectors $u^{(0)}, u^{(1)} \in A$;
3. *There exists $\alpha, \delta > 0$ s.th.*
$$\sup_{u \in A} \|\Psi(u, \eta_1) - \Psi(u, \eta_0)\| \leq \delta \|\eta_1 - \eta_0\|_Y^\alpha$$
for two vectors $\eta_1, \eta_0 \in B$.

Then for every $\eta \in B$, there exists a unique fixed point $u(\eta)$ s.th. $\Psi(u(\eta), \eta) = u(\eta)$. Moreover, there holds
$$\|u(\eta_1) - u(\eta_0)\| \leq \frac{\delta}{1-\rho} \|\eta_1 - \eta_0\|_Y^\alpha . \tag{75}$$

Proof. Fix $u_* \in A$. For every $\eta \in B$, let $u^{(0)}(\eta) := u_*$, $u^{(m)}(\eta) = \Psi(u^{(m-1)}(\eta), \eta)$ for $m = 1, 2, \ldots$, and $u(\eta) := \lim_{m\to\infty} \Psi(u^{(m)}(\eta), \eta)$. This limit exists and is the unique fixed point of Ψ in A, by Item 1-2 in the assumptions and the standard contraction mapping theorem.

For the estimate (75), compute for $m = 1, 2, \ldots$
$$\begin{aligned}
\left\| u^{(m)}(\eta_1) - u^{(m)}(\eta_0) \right\| \leq& \left\| \Psi(u^{(m-1)}(\eta_1), \eta_1) - \Psi(u^{(m-1)}(\eta_0), \eta_1) \right\| \\
&+ \left\| \Psi(u^{(m-1)}(\eta_0), \eta_1) - \Psi(u^{(m-1)}(\eta_0), \eta_0) \right\| \\
\leq& \rho \left\| u^{(m-1)}(\eta_1) - u^{(m-1)}(\eta_0) \right\| + \delta \|\eta_1 - \eta_0\|_Y^\alpha \\
\leq& \delta \sum_{n=0}^{m} \rho^n \|\eta_1 - \eta_0\|_Y^\alpha .
\end{aligned}$$

In the last step we use induction on m and Item 3 in the assumptions. Taking $m \to \infty$ gives (75). □

References

[1] Tobias H. Colding and William P. Minicozzi II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) **175** (2012), no. 2, 755–833. MR2993752

[2] Tobias Holck Colding and William P. Minicozzi II, *Uniqueness of blowups and Lojasiewicz inequalities*, Ann. of Math. (2) **182** (2015), no. 1, 221–285. MR3374960

[3] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature. I*, J. Differential Geom. **33** (1991), no. 3, 635–681. MR1100206

[4] J. Fröhlich, S. Gustafson, B. L. G. Jonsson, and I. M. Sigal, *Solitary wave dynamics in an external potential*, Comm. Math. Phys. **250** (2004), no. 3, 613–642. MR2094474

[5] Zhou Gang, *On the dynamics of formation of generic singularities of mean curvature flow*, 2021.

[6] ———, *On the mean convexity of a space-and-time neighborhood of generic singularities formed by mean curvature flow*, J. Geom. Anal. **31** (2021), no. 10, 9819–9890. MR4303943

[7] Zhou Gang and Dan Knopf, *Universality in mean curvature flow neckpinches*, Duke Math. J. **164** (2015), no. 12, 2341–2406. MR3397388

[8] Zhou Gang, Dan Knopf, and Israel Michael Sigal, *Neckpinch dynamics for asymmetric surfaces evolving by mean curvature flow*, Mem. Amer. Math. Soc. **253** (2018), no. 1210, v+78. MR3803553

[9] Zhou Gang and I. M. Sigal, *Relaxation of solitons in nonlinear Schrödinger equations with potential*, Adv. Math. **216** (2007), no. 2, 443–490. MR2351368

[10] Zhou Gang and Israel Michael Sigal, *Neck pinching dynamics under mean curvature flow*, J. Geom. Anal. **19** (2009), no. 1, 36–80. MR2465296

[11] Mi-Ho Giga, Yoshikazu Giga, and Jürgen Saal, *Nonlinear partial differential equations*, Progress in Nonlinear Differential Equations and their Applications, vol. 79, Birkhäuser Boston, Ltd., Boston, MA, 2010. Asymptotic behavior of solutions and self-similar solutions. MR2656972

[12] Gerhard Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), no. 1, 285–299. MR1030675

[13] W. Schlag, *Stable manifolds for an orbitally unstable nonlinear Schrödinger equation*, Ann. of Math. (2) **169** (2009), no. 1, 139–227. MR2480603

[14] A. Soffer and M. I. Weinstein, *Multichannel nonlinear scattering for nonintegrable equations*, Comm. Math. Phys. **133** (1990), no. 1, 119–146. MR1071238

[15] ———, *Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data*, J. Differential Equations **98** (1992), no. 2, 376–390. MR1170476

[16] Gang Zhou, *On the non-degenerate and degenerate generic singularities formed by mean curvature flow*, 2021.