Qlunch presentation on the adiabatic theory for the area-constrained Willmore flow

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1 Introduction

Let (M, g) be a 3-dimensional, complete, oriented Riemannian manifold with non-negative curvature. Consider the area-constrained Willmore (ACW) flow,

$$\partial_t x^N = -W(x) - \lambda H(x). \tag{1}$$

Here, for $t \geq 0$, $x = x_t : \mathbb{S} \to M$ is a family of embeddings of spheres (with orientation compatible with that on M). $\partial_t x^N$ denotes the normal velocity at x, given by $\partial_t x^N := g(\partial_t x, \nu)$, where $\nu = \nu(x)$ is the unit normal vector to Σ at x. $H(x)$ denotes the mean curvature scalar at x. $W(x) := \Delta H(x) + H(x)(\mathrm{Ric}_M(\nu,\nu) + |\mathring{A}|^2(x))$ is the Willmore operator, where $\mathring{A}(x)$ denotes the traceless part of the second fundamental form. λ is the Lagrange multiplier, arising due to the area constraint.

1.1 Configuration spaces and the geometric structure of ACW

In this subsection, we layout the geometric structure of ACW flow (1).

Let $c \gg 1, k \geq 4$ be given. Define the configuration space

$$X^k := H^k(\mathbb{S}, M), \quad X_c^k := \left\{x \in X^k : |x(\mathbb{S})| = c\right\}. \tag{2}$$

Here, for a surface $\Sigma := x(\mathbb{S}) \subset M$, we denote $|\Sigma| := \int_\Sigma d\mu_\Sigma^g$ the area of Σ, where μ_Σ^g is the area form induced by the embedding x and background metric g. One can check easily that (1) is well-defined in X_c^k. The spaces in (2) are equipped with the L^2-inner product

$$\langle \phi, \phi' \rangle := \int_{\mathbb{S}} \langle \phi, \phi' \rangle_{\mathrm{Euclidean}} \quad (\phi, \phi' \in X^k). \tag{3}$$

Let $x \in X^k$ and write $\Sigma = x(\mathbb{S})$. The tangent spaces to x at X^k and X_c^k are respectively given by

$$T_x X^k = X^k, \tag{4}$$

$$T_x X_c^k = \left\{\phi \in T_x X^k : \int_\Sigma H g(\phi, \nu) = 0\right\}. \tag{5}$$

Here, (5) is due to the well-known first variation formula of the area functional. Notice that, slightly abusing notation, in (5) we view ϕ as a vector field over Σ. With (3), we have a formal Riemannian structure on the configuration spaces X^k and X_c^k.

With this geometric structure of X^k, one can view the equation (1) as the L^2-gradient flow, restricted to X_c^k, of the Willmore energy

$$\mathcal{W}(\Sigma) = \frac{1}{4}\int_\Sigma H^2 \, d\mu_\Sigma^g. \tag{6}$$

We call (the images of) static solutions to (1) *surfaces of Willmore type*, following the nomenclature in [4]. Using Sobolev inequalities, one can show that for $k \geq 4$, the functional $\mathcal{W}$ is well-defined and C^2 (in the sense of Fréchet derivatives) on X_c^k.

Let $d\mathcal{W}(x) : T_x X_c^k \to T_x X_c^{k-4}$ be the Fréchet derivative of $\mathcal{W}$ at an embedding x in the class X_c^k. Define the normal L^2-gradient $\nabla^N \mathcal{W}(x)\phi := d\mathcal{W}(x)\phi$ for every normal, area-preserving variation ϕ on the surface $\Sigma = x(\mathbb{S})$. (This operator ∇^N depends on x.) Then by the first variation formula of the Willmore energy (see e.g. [1, Sec. 3]), this $\nabla^N \mathcal{W}(x)$ is given by the r.h.s. of (1). This allows us to rewrite (1) as

$$\partial_t x^N = \nabla^N \mathcal{W}(x) \quad (x \in X_c^k).$$

Equivalently, (1) is the (negative) gradient flow of the Hawking mass,

$$m_{\mathrm{Haw}}(\Sigma) := \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\left(16\pi - \frac{1}{2}\int_\Sigma H^2 \, d\mu_\Sigma^g\right), \tag{7}$$

in the sense that a flow of surfaces evolving according to (1) increases the mass m_{Haw}. For interests from physics related to this problem, especially in general relativity, see [5].

1.2 Main result

Under suitable assumptions on the background manifold, we derive the following results in [6]:

Theorem 1 (Main)**.** *Let $k \geq 4$, $c \gg 1$. Let X^k be the configuration space defined in* (2)*. Fix $R \gg 1, \delta \ll 1$ in Definition* **??***.*

Then there exists a map

$$\tilde{\Phi} : M' := \mathbb{R}_{>R} \times B_1(0) \subset \mathbb{R} \times \mathbb{R}^3 \to X^k$$

with the following property: Let $x_{r,z} := \tilde{\Phi}(r, z)$. There hold:

1. *(Critical point) x_z parametrizes a surface of Willmore type if and only if z is a critical point of the function $\mathcal{W} \circ \tilde{\Phi} : \mathbb{R}^4 \to \mathbb{R}$, restricted to the submanifold $\left\{(r, z) \in M' : \left|\tilde{\Phi}(r, z)(\mathbb{S})\right| = c\right\}$.*
2. *(Stability) Suppose x_z parametrize an admissible surface of Willmore type. Then x_z is uniformly stable with small area-preserving H^k-perturbation* [1] *if z is a strict local minimum of the function $\mathcal{W} \circ \tilde{\Phi}$ restricted to the submanifold $\left\{(r, z) \in M' : \left|\tilde{\Phi}(r, z)(\mathbb{S})\right| = c\right\}$.*
3. *(Effective dynamics) Let $\Sigma_* = x_*(\mathbb{S})$ be an admissible surface. Let $\Sigma_t = x_t(\mathbb{S})$ be the global solution to* (1) *with initial configuration $\Sigma|_{t=0} = \Sigma_*$ as in Theorem* **??***.*

   Then there exist $\alpha > 0$, $T = O(R^{-\alpha})$, and a path $(r_t, z_t) \in M'$, such that for every $t \geq T$,

   $$\|\tilde{\Phi}(r_t, z_t) - x_t\|_{X^k} = O(R^{-3}). \tag{8}$$

   Moreover, the path (r_t, z_t) evolves according to

   $$\dot{z} = \frac{1}{4\pi}\nabla_z \mathcal{W} \circ \tilde{\Phi}(r, z) + O(R^{-3}), \tag{9}$$

   $$\dot{r} = 4R^{-2} + O(R^{-3}). \tag{10}$$

   In (9) *the leading term is of the order $O(R^{-2})$.*
4. *Conversely, if $(r_t, z_t) \in M'$ is a flow evolving according to* (9)*-*(10)*, then there exists a global solution x_t to* (1) *such that* (8) *holds for this choice of (r_t, z_t) and every $T \leq t \leq T + R$.*

2 The Lyapunov-Schmidt map

Let $k \geq 4, c \gg 1$. Let $K \subset M$, $R \gg 0$ to be determined, and let $M' := \mathbb{R}_{>R} \times B_1(0) \subset \mathbb{R} \times \mathbb{R}^3$. In this section we construct the map $\tilde{\Phi} : M' \to X^k$ as in Theorem 1.

2.1 Graphs over sphere

Denote $H^k = H^k(\mathbb{S}, \mathbb{R})$. This space is equipped with the L^2-inner product $\langle u, v\rangle = \int_{\mathbb{S}} uv$. Define the configuration space

$$Y^k := H^k \times M'. \tag{11}$$

Define a map

$$\begin{array}{rccl} \theta & : & Y^k & \longrightarrow & X^k \\ & & (\phi, r, z) & \longmapsto & r(1 + \phi(v))v + z \end{array}. \tag{12}$$

Here $v \in \mathbb{S} \subset \mathbb{R}^3$ is the spherical coordinate, and recall we identify the asymptotic part $(M \setminus K) \cong (\mathbb{R}^3 \setminus B_R(0))$. Define

$$Y^k_c := \left\{(\phi, r, z) \in Y^k : \theta(\phi, r, z) = c\right\} \tag{13}$$

This corresponds to the space of surfaces with fixed area, X^k_c, as in (2).

For $\|\phi\|_{H^k} \ll 1$, the map $\theta(\phi, r, z)$ is a well-defined graph over the coordinate sphere $\theta(0, r, z)(\mathbb{S}) =: S_{r,z}$. Thus we can also identify $\theta(\phi, r, z)$ as a function from $S_{r,z} \subset M \to \mathbb{R}$. Note also that for sufficiently large $c \gg 1$ and every $z \in B_1(0) \subset \mathbb{R}^3$, there is a coordinate sphere with area c around z. Thus the map θ is surjective onto X^k_c.

Definition 1 (topology on graphs)**.** We say two graphs $\theta(\phi, r, z)$, $\theta(\phi', r', z')$ are H^k-close if $\|\phi - \phi'\|_{H^k} + |r - r'| + |z - z'| \ll 1$.

[1] This means that if y is another admissible surface that is H^k-close to x_z, then for every $\epsilon > 0$ there exists $T > 0$ such that $\|x_t - y_t\|_{X^k} < \epsilon$ for all $t \geq T$, where x_t, y_t are respectively the flows generated by x_z, y under (1).

2.2 Lyapunov-Schmidt reduction

Denote $\bar{W}(\phi, r, z)$, $\Omega(\phi, r, z)$ the pullbacks of the r.h.s. of (1) and the Willmore energy (6) to Y^k through θ, respectively. Explicitly, we have

$$\bar{W}(\phi, r, z) := -W(\theta(\phi, r, z)) - \lambda H(\theta(\phi, r, z)), \tag{14}$$

$$\Omega(\phi, r, z) := \mathcal{W}(\theta(\phi, r, z)). \tag{15}$$

Since $\mathcal{W}$ is C^2 on X^k with $k \geq 4$ and θ is smooth, the pullback energy Ω is C^2 on Y^k, $k \geq 4$. Using Sobolev inequalities, one can check that the partial Fréchet derivative $\bar{W}$ is C^1 in ϕ and smooth in r, z. This $\bar{W}$ is the L^2-gradient of $\Omega(\cdot, r, z)$ up to scaling, and satisfies the mapping property $\bar{W} : Y^k \to H^{k-4}$.

Remark 1. Notice that (14)-(15) both depend implicitly on the background metric g.

Lemma 1. *The linearized operator $L_{r,z}$ of $\bar{W}$ at $(0, r, z)$ with background metric g is given by*

$$\begin{aligned} L^g_{r,z} :&= \partial_\phi \bar{W}(\phi, r, z)|_{\phi=0} \\ &= (\Delta^2 + 2r^{-2}\Delta + O(r^{-4}))\partial_\phi \theta(0, r, z) : H^k \to H^{k-4}. \end{aligned} \tag{16}$$

Here $\Delta : X^k \to X^{k-2}$ denotes the Laplace-Beltrami operator on the coordinate sphere $S_{r,z} \subset M \setminus K$, with center z and radius r. The partial Fréchet derivative $\partial_\phi \theta(0, r, z) : H^k \to X^k$ is given by $\xi(v) \mapsto \xi(v)rv$.

Moreover, the operator $L_{r,z}$ is self-adjoint on H^k. The spectrum of $L_{r,z}$ is purely discrete. The operator $\partial_\phi \theta(0, r, z)$ is invertible and satisfies

$$\|\partial_\phi \theta(0, r, z)\|_{H^k \to X^k} = \|\partial_\phi \theta(0, r, z)^{-1}\|^{-1}_{X^k \to H^k} = r. \tag{17}$$

Proof. The operator $L^g_{r,z}$ is explicitly calculated in [4, Sec. 3]. The spectral properties of $L_{r,z}$ are studied in [4, Sec. 7]. The mapping properties of $\partial_\phi \theta$ is obvious. □

Remark 2. The linearized operator (16) depends on (the curvature of) the background metric g on M. In the special case when the ambient manifold M is flat, i.e. $g = \delta_{ij}$, the linearized operator $L^0_{r,z}$ has eigenvalue 0, and $\ker L^0_{r,z}$ is spanned by the constant function $y^0 \equiv 1$, together with the spherical harmonics y^1, y^2, y^3. Thus, so long as (M, g) is asymptotically flat and $r \gg 1$ in (16) (such as in our setting), one can view $L^g_{r,z}$ as a perturbation of $L^0_{r,z}$. This motivates the following definition.

Definition 2. Define $P : H^k \to H^k$ to be the L^2-orthogonal projection onto span $\{y^0, \dots, y^4\} = \ker L^0_{r,z}$. Define $\bar{P} := 1 - P : H^k \to H^k$ be the complement of P.

Let $\mathcal{S}$ be the set of all smooth symmetric two tensors on M. Define a map

$$\begin{array}{cccc} F & : & Y^k \times \mathcal{S} & \longrightarrow & H^{k-4} \\ & & (\phi, r, z, h) & \longmapsto & \bar{P}\bar{W}(\phi, r, z) \end{array}, \tag{18}$$

where $\bar{W}$ is computed with background metric $g = g_S + h$ (see Section **??**).

Proposition 1. *Assume the ambient manifold (M, g) is C^k-closed to Schwarzschild.*

1. *For every $z \in \mathbb{R}^3$ with $|z| < 1$ and sufficiently large $r \geq R \gg 1$, there is a unique solution $\phi = \phi_{r,z} \in \bar{P}H^k$ to the equation*
$$F(\phi, r, z, h) = 0, \tag{19}$$
where F is defined in (18)*, and $g = g_S + h$.*

2. *Moreover, the map $(r, z) \mapsto \phi_{r,z}$ is C^2, and satisfies the estimate*
$$\|\partial^m_r \partial^\alpha_z \phi_{r,z}\|_{H^k} \lesssim r^{-(2+2m)}. \tag{20}$$
for every $m + |\alpha| \leq 2$.

3. *Moreover, the surface $\theta(\phi_{r,z}, r, z)$ lies in the class of admissible surfaces in Definition* **??**

Proof. 1. By the Implicit Function Theorem, it suffices to check that the map F defined in (19) satisfies the following properties:

1. F is C^1 in ϕ.
2. $F(0, r, z, 0) = 0$ for every r, z.
3. $\partial_\phi F(0, r, z, 0) = L^0_{r,z}$ is invertible on $\bar{P}H^k$.

The first claim follows from the regularity of $\bar{W}$ on Y^k and its smooth dependence on the background metric.

If the background metric is Schwarzschild, i.e. $h = 0$, then it is well-known that by conformal invariance the coordinate sphere $\theta(0, r, z)$ is the global minimizer of the Willmore energy $\mathcal{W}$. Since $\bar{W} = \partial_\phi \Omega$ (see (15)), the second claim follows.

The spectrum of $L^0_{r,z}$ can be calculated explicitly. See for instance [2, Cor. 33]. In particular, 0 is an isolated eigenvalue with finite multiplicity. By elementary spectral theory, this implies the restriction $\bar{L}^0_{r,z} := L^0_{r,z}|_{\bar{P}}$ is invertible as a map from $\bar{P}H^k \to \bar{P}H^k$. Thus the third claim follows.

2. For the estimate (20), we expand

$$L^g_{r,z} = L^0_{r,z} + V_{r,z}, \tag{21}$$

where $V_{r,z}$ is defined by this expression. As we discuss in Remark 2, this $V_{r,z}$ is bounded from $H^k \to H^{k-4}$, and satisfies $\|V_{r,z}\|_{H^k \to H^{k-4}} = O(r^{-4})$. The restriction $\bar{L}^0_{r,z}$ can bounded from below by Cr^{-2} for some $C > 0$ only depending on k. It follows that

$$\|(\bar{L}^0_{r,z})^{-1} V_{r,z}\|_{H^{k-4} \to H^k} = O(r^{-2}).$$

For sufficiently large r, this together with the expansion (21) implies that the restriction $\bar{L}^g_{r,z} : \bar{P}H^k \to \bar{P}H^{k-4}$ is invertible, given explicitly as the Neumann series

$$(\bar{L}^g_{r,z})^{-1} = \sum_{n=0}^{\infty} (\bar{L}^0_{r,z})^{-1} (-V_{r,z} (\bar{L}^0_{r,z})^{-1})^n.$$

From here one can also read off the estimate

$$\|(\bar{L}^g_{r,z})^{-1}\|_{\bar{P}H^{k-4} \to \bar{P}H^k} = O(r^2). \tag{22}$$

Expand $F(\phi, r, z, h) = F(0, r, z, h) + \bar{L}^g_{r,z}\phi + N_{r,z}(\phi)$, where the nonlinearity $N_{r,z}$ is defined by this expression. This $N_{r,z}$ is calculated explicitly in (**??**). For every ϕ satisfying (19), we can rearrange to get

$$\phi = -(\bar{L}^g_{r,z})^{-1}(F(0, r, z, h) + N_{r,z}(\phi)). \tag{23}$$

In the r.h.s. we have $F(0, r, z, h) = O(r^{-4})$ by [2, Cor. 45]. Thus, for sufficiently small ϕ, we have by (22)-(23) that $\|\phi\|_{H^k} = O(r^{-2})$.

We now claim for $\phi \in H^k$ and $m + |\alpha| \le 2$, there hold

$$\|\partial_r^m \partial_z^\alpha (\bar{L}^g_{r,z})^{-1} \phi\|_{H^k} \lesssim \|\phi\|_{H^{k-4}}, \tag{24}$$

$$\|\partial_r^m \partial_z^\alpha F(0, r, z, h)\|_{H^{k-4}} \lesssim r^{-(4+m)}, \tag{25}$$

$$\|\partial_r^m \partial_z^\alpha N_{r,z}(\phi)\|_{H^{k-4}} \lesssim \|\phi\|^2_{H^k}. \tag{26}$$

For (24), one uses the identity $\partial^\beta (\bar{L}^g_{r,z})^{-1} = -(\bar{L}^g_{r,z})^{-1} \bar{\partial}^\beta L^g_{r,z} (\bar{L}^g_{r,z})^{-1}$, where $|\beta| \le 2$ is a multi-index in both r and z. This, together with the fact that $\partial^\beta L_{r,z}$ is uniformly bounded (see (**??**)), implies (24). The rest follows from the expansion in Proposition **??**. Using (24)-(26) and differentiating both sides of (23), we conclude the estimates (20).

3. For sufficiently large R and every $r \ge R$, we find using (20) with $m = 0, \alpha = 0$ that the surface $\theta(\phi_{r,z}, r, z)$ is H^k-close to the coordinate sphere $S_{r,z}$. This implies $\theta(\phi_{r,z}, r, z)$ is an admissible surface.

□

From now on we write $\zeta = \zeta^\alpha$, $\alpha = 0, \dots, 4$, for a point in $(r, z) \in M'$. Thus, $\zeta^0 = r$ and $\zeta^j = z^j$ for $j = 1, 2, 3$.

Definition 3 (The Lyapunov-Schmidt map Φ)**.** Let $K \subset M$ be the compact set as in Theorem **??**. Let $R \gg 1$, $\delta \ll 1$ be given as in Theorem **??**. Let $M' := \mathbb{R}_{>R} \times B_1(0) \subset \mathbb{R} \times \mathbb{R}^3$.

Define *the Lyapunov-Schmidt map* $\Phi : M' \to H^k$ by $\zeta \mapsto \phi_\zeta$, where ϕ_ζ is the solution to (19) given in Proposition 1.

Remark 3. This Φ is equivalent to the map $\tilde{\Phi}$ in Theorem 1, through the diffeomorphism $\Phi(\zeta) \mapsto \theta(\Phi(\zeta), \zeta)$.

In the next proposition, we describe the geometric structure induced by the map Φ.

Proposition 2. *The set*

$$E := \{\theta(\phi, \zeta) : \phi = \Phi(\zeta), \zeta \in M'\}$$

forms an immersed C^1 submanifold in X^k. The tangent space $T_{\theta(\Phi(\zeta),\zeta)}E$ consists of vector fields over the surface $\theta(\Phi(\zeta), \zeta)(\mathbb{S})$. A basis of $T_{\theta(\Phi(\zeta),\zeta)}E$ is given by $\partial_{\zeta^\alpha}\theta(\Phi(\zeta), \zeta)$.

Remark 4. Using the projection constructed in Lemma 2, one can view this manifold E as consisting of the adiabatic parts of low (Willmore) energy surfaces in X^k.

Proof. The manifold structure of E follows from Definition 3, where $\Phi : M' \to E$ is a C^1 parametrization. We check the tangent space is non-degenerate. Compute

$$\partial_{\zeta^0}\theta(\Phi(\zeta), \zeta)(v) = (1 + \Phi(\zeta) + \zeta^0 \partial_{\zeta^0}\Phi(\zeta))v, \tag{27}$$

$$\partial_{\zeta^j}\theta(\Phi(\zeta), \zeta)(v) = \zeta^0 \partial_{\zeta^j}\Phi(\zeta)v + e^j, \tag{28}$$

where e^j is the j-th unit vector in $\mathbb{R}^3$. By the estimate (20), we find

$$\langle \partial_{\zeta^\alpha}\theta(\Phi(\zeta), \zeta), \partial_{\zeta^\beta}\theta(\Phi(\zeta), \zeta) \rangle = 4\pi\delta_{\alpha\beta} + O(R^{-2}).$$

This implies the claim if R is sufficiently large. □

In Appendix, we introduce the general concepts of the Lyapunov-Schmidt map, and relate it to our setting above.

2.3 Barycenter

In this subsection, we develop a new concept of barycenter for a certain class of closed surfaces in X^k.

Definition 4 (Barycenter). Let x_* be an embedding of sphere that is H^k-close to the manifold $E \subset X^k$ constructed in Definition 3, w.r.t. the topology on graphs introduced in Definition 1. Then we can write $x_* = \theta(\Phi(\zeta) + \xi, \zeta)$ for some $\zeta \in M', \|\xi\|_{H^k} \ll 1$. (There can in general be many such choice of ζ and ξ.) Expand x_* in ξ around $\theta(\Phi(\zeta), \zeta)$ as

$$x_* = \theta(\Phi(\zeta), \zeta) + \partial_\phi \theta(\Phi(\zeta), \zeta)\xi + O(\|\xi\|^2_{H^k}). \tag{29}$$

Define $f_\alpha \in H^k$ as

$$\begin{aligned} f_\alpha(\zeta)(v) &= \partial_{\zeta^\alpha} \theta(\Phi(\zeta), \zeta)(v)^N \\ &= g(\partial_{\zeta^\alpha} \theta(\Phi(\zeta), \zeta)), \nu(\theta(\Phi(\zeta), \zeta)) \quad (\alpha = 0, \dots, 3). \end{aligned} \tag{30}$$

We say a point $\zeta_* \in M'$ is the barycenter of x_* if ζ_* solves the following algebraic system:

$$\langle \xi, f_\alpha \rangle_{L^2} = 0 \quad (\alpha = 0, \dots, 3), \tag{31}$$

where ξ is defined by the relation $x_* = \theta(\Phi(\zeta_*) + \xi, \zeta_*)$.

Remark 5. The four vectors f_α span the tangent space at $\theta(\Phi(\zeta_t), \zeta_t)^N$ to $E^N \subset H^k$, where E^N consists of the normal components of the elements in the manifold E defined in Definition 3. Geometrically, the defining condition (31) for barycenter means that the Gâteaux derivative of the map $\theta(\cdot, \zeta_*)^N$ at $\Phi(\zeta_*)$ along ξ-direction is perpendicular to the tangent space $T_{\theta(\Phi(\zeta_*), z_*)^N} E^N$. In terms of the expansion (29), this means the second term in the r.h.s. is L^2-orthogonal to the tangent space at the first term to E. In this sense the choice of barycenter is optimal.

Remark 6. Our definition of barycenter differs from the classical one, given by averaging over Σ w.r.t. Euclidean background metric, namely $|\Sigma|_g^{-1} \int_\Sigma x d\mu_\Sigma^{\delta_{ij}}$. See [3] and the references therein. Our version of barycenter retains the key decay property as [3, Sec. 5]. Namely, the motion of barycenter is controlled by a differential inequality using a Lyapunov functional, defined in Section **??**.

Moreover, our definition allows us to retain explicit and uniform control of a flow evolving according to (1), as we show in Sec. **??**.

In the next lemma, we define a nonlinear projection (or coordinate map) that associates barycenters to low energy configurations in X^k.

Lemma 2 (nonlinear projection). *There exists $\delta > 0$ such that on the space*

$$U_\delta := \{x = \theta(\Phi(\zeta) + \xi, \zeta) : \zeta \in M', \|\xi\|_{H^k} < \delta\}, \tag{32}$$

there exists a C^1 map $S : U_\delta \to M'$ such that $S(x)$ is the barycenter of x as in Definition 4.

Moreover, we have uniform estimate on S and its derivative.

Remark 7. Essentially, the existence of such projection depends on the non-degeneracy shown in Proposition 2. Later, we see that the barycenter $\zeta = S(x)$ determines the adiabatic (or slowly-varying) part of x.

We call the remainder ξ that satisfies $x = \theta(\Phi(S(x)) + \xi, S(x))$ the *fluctuation* of x.

Proof. Define a map

$$\begin{array}{cccccc} \Gamma & : & M' \times U_\delta \subset \mathbb{R}^4 \times H^k & \longrightarrow & \mathbb{R}^4 \\ & & (\zeta, x) & \longmapsto & \langle \xi, f_\alpha \rangle_{L^2} \end{array}, \tag{33}$$

where ξ is defined by the relation $x = \theta(\Phi(\zeta) + \xi, \zeta)$. It suffices to find a map S such that $\zeta = S(x)$ solves

$$\Gamma(\zeta, x) = 0. \tag{34}$$

By the Implicit Function Theorem, it suffices to check that the map Γ satisfies the following properties:

1. Γ is C^1 in ζ.
2. $\Gamma(\zeta, \theta(\Phi(\zeta), \zeta)) = 0$.
3. The matrix $\nabla_\zeta \Gamma(\zeta, \theta(\Phi(\zeta), \zeta)) : \mathbb{R}^4 \to \mathbb{R}^4$ is invertible.

The first claim follows from the regularity of Φ as in Proposition 1. The second claim is trivial because in this case $\xi = 0$.

We now claim the rescaled matrix

$$A_{\alpha\beta} := \zeta^0 \partial_{\zeta^\alpha} \Gamma_\beta|_{(\theta(\Phi(\zeta), \zeta))}$$

is invertible, which implies the third claim above.

Using (27)-(28), the definition (30), and the assumption on the background metric $g_{ij} = \delta_{ij} + O((\zeta^0)^{-1})$, we compute

$$f_0 = (1 + O((\zeta^0)^{-1}))y^0 + \Phi(\zeta) + \zeta^0 \partial_{\zeta^0} \Phi(\zeta) + \Phi(\zeta) O((\zeta^0)^{-1}) + O((\zeta^0)^{-4}), \tag{35}$$

$$f_j = (1 + O((\zeta^0)^{-1}))y^i + \zeta^0 \partial_{\zeta^j} \Phi(\zeta) + O((\zeta^0)^{-4}), \tag{36}$$

where the vectors y^α span $\operatorname{ran} P$ as in Definition 2.

For $\xi = \xi(\zeta, x)$, we find

$$\xi(v) = \left\langle \frac{x(v) - \zeta}{\zeta^0},\, v \right\rangle - 1 - \Phi(\zeta)(v), \tag{37}$$

$$\partial_{\zeta^0}\xi(v) = -\left\langle \frac{x(v) - \zeta}{(\zeta^0)^2},\, v \right\rangle - \partial_{\zeta^0}\Phi(\zeta), \tag{38}$$

$$\partial_{\zeta^j}\xi(v) = -\frac{e^j}{\zeta^0} - \partial_{\zeta^j}\Phi(\zeta)(v). \tag{39}$$

Now, since $x \in U_\delta$, we have $A_{\alpha\beta} = \zeta^0 \langle \partial_{\zeta^\alpha}\xi,\, f_\beta\rangle + \zeta^0 \langle \xi,\, \partial_{\zeta^\alpha} f_\beta\rangle = \langle \partial_{\zeta^\alpha}\xi,\, f_\beta\rangle + O(\zeta^0\delta)$. By this, and the formula (35)-(39) above, we find that $A_{\alpha\beta} = O(1)\delta_{\alpha\beta} + O((\zeta^0)^{-2}) + O(\zeta^0\delta)$. For sufficiently large $R \gg 1$, $\delta = o(R^{-1})$, and every $\zeta^0 \geq R$, we can conclude from here that $A_{\alpha\beta}$ is invertible. This proves the existence of the C^1 map S. The uniform estimates for S and its Fréchet derivative are implicit in the arguments above.

□

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