Asymptotic Stability of Cylindrical Singularities

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Set up

Consider the mean curvature flow (MCF) for a family of hypersurfaces given by immersions

$$X(\cdot, t) : \mathbb{R}^{n-k} \times \mathbb{R}^{k+1} \to \mathbb{R}^{n+1}, \quad 0 \leq t < T,$$

satisfying

$$\partial_t X = -H(X)\nu(X). \tag{MCF}$$

We are interested in the dynamical behaviour of a solution X to (MCF), which first develops a singularity at $0 \in \mathbb{R}^{n+1}$, $t = T > 0$.

Rescaling

We consider the following time-dependent rescaling for a solution $X(\cdot, t) : \mathbb{R}^{n-k}_x \times \mathbb{R}^{k+1}_\omega \to \mathbb{R}^{n+1}$ to (MCF) as follows:

$$X(x, \omega, t) = \underbrace{\lambda(t)}_{\in \mathbb{R}_{>0}} \underbrace{g(t)}_{\in O(n+1)} Y(\underbrace{y(x,t)}_{\in \mathbb{R}^{n-k}}, \omega, \underbrace{\tau}_{\in \mathbb{R}_{\geq 0}}) + \underbrace{\zeta(t)}_{\in \mathbb{R}^{n+1}}, \qquad \text{(R)}$$

Here, the immersion Y is defined through this relation, and

$$a(t) \in \mathbb{R}_{>0}, \quad \lambda(t) := \left(2 \int_t^T a(t')\, dt' \right)^{1/2},$$

$$y(x, t) := \lambda(t)^{-1} x, \quad \tau(t) := \int_0^t \lambda(t')^{-2}\, dt'.$$

Remarks. Consider $X(g, \zeta, a, Y) = \lambda(a) g Y(y, \omega, \tau) + \zeta$ in (R).

1. $\lambda = \lambda(t)$ is uniquely determined by the function $a = a(t) > 0$. Indeed, this λ is the unique solution to the Cauchy problem $\lambda \partial_t \lambda = -a, \lambda(T) = 0$.
2. The terminal condition on λ ensures that the rescaling (R) gives rise to a tangent flow $Y = Y(y, \omega, \tau)$ in the microscopic variable $y = \lambda^{-1} x$ and slow time variable $\tau = \int^t \lambda^{-2}(t')\, dt'$.
3. We view (g, ζ, a) in the rescaling (R) as an unknown a path in

$$(g, \zeta, a) \in \Sigma := O(n+1) \times \mathbb{R}^{n+1} \times \mathbb{R}_{>0}.$$

Rescaled MCF

Plugging (R) into (MCF), we find that $X = X(g, \zeta, a, Y)$ solves the MCF if and only if the quadruple (g, ζ, a, Y) solves

$$\partial_\tau Y = -H(Y)\nu(Y) - a\left\langle y,\, \nabla_y\right\rangle Y + aY - g^{-1}\partial_\tau g Y - \lambda^{-1} g^{-1}\partial_\tau \zeta.$$

Call this *the rescaled mean curvature flow.*

Stationary solutions (cylinders in $\mathbb{R}^{n+1}$):

$$Y \equiv Y_{a_0} := \left(y, \sqrt{\frac{k}{a_0}}\omega\right), \tag{1}$$

$$(g, \zeta, a) \equiv (g_0, \zeta_0, a_0) \in \Sigma. \tag{2}$$

Graphical equations

We seek maximal solution X to MCF on $\mathbb{R}^{n-k}_x \times \mathbb{S}^k_\omega \times \mathbb{R}_{0\le t<T}$ of the form (c.f. (R))

$$X(x,\omega,t) = \lambda(t)g(t)\underbrace{\left(y(x,t), \left(\sqrt{\frac{k}{a(t)}} + \xi(y(x,t),\omega,\tau(t))\right)\omega\right)}_{\text{normal perturbation of the stationary sol. to RMCF}} + \zeta(t)$$

Here and below, we write X of this form as $X = X(\sigma, \xi)$, where

1. $\sigma \equiv (g, \zeta, a)$ is a path of symmetry.
2. $\xi : \mathbb{R}^{n-k}_y \times \mathbb{R}^{k+1}_\omega \times \mathbb{R}_{\tau\ge 0} \to \mathbb{R}$ is a small (normal) perturbation.

Configuration spaces

For $s \geq 0, a > 0$, define the Gaussian weighted Sobolev space

$$X^s(a) := H^s(\mathbb{R}_y^{n-k} \times \mathbb{S}_\omega^k, \mathbb{R};\, \rho_a), \quad \rho_a := e^{-a|y|^2/2}\, d\mu. \qquad (3)$$

Here $d\mu$ is the canonical measure on $\mathbb{R}^{n-k} \times \mathbb{S}^k$.

For $s \leq r$, $0 < b \leq a$, clearly, $X^r(b) \subset X^s(a)$.

Huisken's F-functional:

$$F_a(v) := \int_S \rho_a\, d\mu_S, \quad S := \left\{ (y, v(y,\omega)\omega) : v : \mathbb{R}_y^{n-k} \times \mathbb{R}_\omega^{k+1} \to \mathbb{R} \right\}. \qquad (4)$$

This is C^2 on $X^s(a)$ with $a > 0$, $s \geq 2$ (assume this from now on).

Lemma (Implied Equation)

$X = X(\sigma, \xi)$ solves the MCF if and only if (σ, ξ) satisfy

$$\dot{\xi} = -F_a'(\sqrt{k/a} + \xi) - \partial_\sigma W(\sigma)\dot{\sigma}, \tag{5}$$

Here,

$F_a'(v)$ *is the* $X^0(a)$*-gradient of* F_a *at* v*,*

$W : \Sigma \ni \sigma \mapsto \sqrt{k/a} + g_{n-k+l,j}\omega^l y^j + \langle z,\, \lambda^{-1}\omega \rangle \in X^s(a).$

Proof.

Direct computation by plugging $X = X(\sigma, \xi)$ into the RMCF. □

Below we call (5) the *graphical RMCF*.

Main Theorem: Set up

Let $X^s(a)$, $s \geq 2$, $a > 0$ be the Gaussian weighted Sobolev space.There exists $0 < \delta \ll 1$ s.th. the following holds: For every $a_0 \geq 1/2 + 2\delta$, there exists a linear subspace $\mathcal{S} \subset X^s(a_0)$ with finite codimensions, an open set $\mathcal{B}_\delta \subset \{\|\eta\|_{X^s} < \delta\}$, and a map

$$\Phi : \mathcal{B}_\delta \cap \mathcal{S} \to X^s \equiv X^s(1/2), \quad \text{satisfying}$$

$$\|\Phi(\eta_0)\|_{X^s} \lesssim \|\eta_0\|_{X^s}^2 \quad \text{(quadratic)},$$

$$\|\Phi(\eta_0) - \Phi(\eta_1)\|_{X^s} \lesssim \delta \, \|\eta_0 - \eta_1\|_{X^s} \quad \text{(Lipshitz)},$$

for every η_0, $\eta_1 \in \mathcal{B}_\delta \cap \mathcal{S}$, as well as the following properties:

Main Theorem: Global existence

For every $\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S}$, there exists a global (i.e. $0 \leq \tau < \infty$) solution $(\sigma = (g, \zeta, a), \xi)$ to the graphical rescaled MCF

$$\dot{\xi} = -F_a'(\sqrt{k/a} + \xi) - \partial_\sigma W(\sigma)\dot{\sigma},$$

with initial configuration

$$\xi|_{\tau=0} = \eta_0 + \Phi(\eta_0), \quad \sigma|_{t=0} = (\mathbf{1}_{n+1}, 0, a_0).$$

By Implied Equation Lemma, this gives rise to a maximal sol. to MCF on $\mathbb{R}^{n-k} \times \mathbb{S}^{k+1} \times \mathbb{R}_{0 \leq t < T}$, namely $X = X(\sigma, \xi)$.

Main Theorem: Dissipative estimates

Here and below, we write

$$\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2}.$$

The solution $\xi(\cdot, \tau)$ from the existence part is non-negative for all τ, and dissipates to zero, with the decay estimate

$$\|\xi(\cdot, \tau)\|_{X^s} \leq \delta \langle \tau \rangle^{-2}, \quad \tau \geq 0. \tag{6}$$

In fact, the choice of the open set $\mathcal{B}_\delta \subset \{\|\eta\|_{X^s} < \delta\}$ ensures $\xi(\cdot, \tau) \geq 0$ for all τ, which guarantees embeddedness of $X(\sigma, \xi)$.

Remarks on the Main Theorem

1. By definition, up to a rigid motion, cylinders with radius $\sqrt{k/a_0}$ correspond to the following stationary solution to the graphic RMCF

$$\sigma_0 \equiv (g_0, \zeta_0, a_0), \quad \xi_0 \equiv 0.$$

2. By the Main Theorem, the set

$$M := \{\eta + \Phi(\eta) : \eta \in \mathcal{S} \cap \mathcal{B}_\delta\}$$

forms a non-degenerate, finite codimensional stable manifold for the graphic RMCF, parametrized by $\mathcal{S} \subset X^s(a_0)$.

Typical element in M

Stability of Cylindrical Singularities

- In [Ann. of Math. (2) 175 (2012)], Colding-Minicozzi showed cylindrical singularities are F-unstable.
- In terms of the graphical RMCF, this means that the static solution $\xi_0 \equiv 0$ is *linearly unstable.*
- By Main Theorem above, under a generic class of initial perturbations, namely those in the finite-codimensional stable manifold M, the static sol. ξ_0 is actually *asymptotically stable*: a generic perturbation $\xi = \xi_0 + \eta + \Phi(\eta)$ dissipates to $\xi_0 = 0$ as $\tau \to \infty$.

Recall $F_a : X^s(a) \to \mathbb{R}$ (Huisken's F-functional) is a C^2 functional. The linearized operator of $F_a'(v)$ at the critical point $v \equiv \sqrt{k/a}$ is

$$L(a) = -\Delta_y + a \langle y,\, \nabla_y(\cdot)\rangle - \frac{a}{k}\Delta_\omega - 2a.$$

Here $a > 0$ corresponds to the cylindrical radius. From now on we write the graphical RMCF as

$$\begin{aligned} \dot{\xi} &= -\, F_a'(\sqrt{k/a} + \xi) - \partial_\sigma W(\sigma)\dot{\sigma} \\ &= -\, L(a)\xi - N(a, \xi) - \partial_\sigma W(\sigma)\dot{\sigma}. \end{aligned}$$

Here $N(a, \xi) := F_a'(\sqrt{k/a} + \xi) - L(a)\xi$ is the nonlinearity.

Linearized operator

The fact that cylinders are F-unstable has to do with the linearized operator at the cylinder.

Lemma (Colding-Minicozzi)

The linearized operator

$$L(a) = -\Delta_y + a\,\langle y,\, \nabla_y(\cdot)\rangle - \frac{a}{k}\Delta_\omega - 2a$$

is self-adjoint in $X^s(a)$*, and is bounded from* $X^s(a) \to X^{s-2}(a)$*. The spectrum of* $L(a)$ *is purely discrete, and the only non-positive eigenvalues, together with the associated eigenfuncitons, are*

Zero-unstable modes of $L(a)$

$$-2a, \quad \text{with eigenfunction } \Sigma^{(0,0)(0,0,0)}(a) := -\frac{\sqrt{k}}{2} a^{-3/2},$$

$$-a, \quad \text{with eigenfunctions } \Sigma^{(0,1)(0,0,l)}(a) := \lambda^{-1}\omega,$$

$$-a, \quad \text{with eigenfunctions } \Sigma^{(1,0)(i,0,0)}(a) := \frac{1}{\|y^i\|_{0,a}^2} y^i,$$

$$0, \quad \text{with eigenfunctions } \Sigma^{(1,1)(i,0,l)}(a) := y^i \omega^l,$$

$$0, \quad \text{with eigenfunctions } \Sigma^{(2,0)(i,j,0)}(a) := \frac{1}{\|a y^i y^j - \delta_{ij}\|_{0,a}^2}(a y^i y^j - \delta_{ij}).$$

Remark. Some, but not all of the zero-unstable modes of the linearized operator $L(a)$ are due to broken symmetries.

Main ideas

- So far we have only one equation for ξ, whereas we are solving for a pair of unknowns (σ, ξ).
- Introduce an equation for σ (*the modulation equation*) to remove the effect of the symmetry zero-unstable modes.
- Incorporate certain *zero-unstable modes* into the solution (!) to ensure dissipative estimates at $\tau \to \infty$.

The last point was first rigorously implemented in [J. Geom. Anal. 19 (2009)] by Zhou Gang and Sigal. Similar modulation method is customary in the study of e.g. NLS soliton.

Modulation equations

Recall the following zero-unstable modes of linearized opr. $L(a)$:

$$-2a, \quad \text{with eigenfunction } \Sigma^{(0,0)(0,0,0)}(a) := -\frac{\sqrt{k}}{2}a^{-3/2},$$

$$-a, \quad \text{with eigenfunctions } \Sigma^{(0,1)(0,0,l)}(a) := \lambda^{-1}\omega,$$

$$0, \quad \text{with eigenfunctions } \Sigma^{(1,1)(i,0,l)}(a) := y^i\omega^l,$$

For these $\Sigma^{(m,n)}(a)$ with $(m,n) = (0,0),(0,1),(1,1)$, there exists a path $\sigma(\tau) \in \Sigma$ (the symmetry Lie group of MCF) s.th. we can eliminate the distablizing effect of these modes.

Lemma (Modulation)

Suppose (σ, ξ) *is a global solution to the graphical RMCF s.th.*

$$\left\langle \xi(0),\, \Sigma^{(m,n)}(a(0)) \right\rangle_{a(0)} = 0 \quad (\langle \cdot,\, \cdot \rangle_a = \textit{inn. prod. on } X^0(a))$$

for $(m, n) = (0, 0), (0, 1), (1, 1)$.

Then ξ *satisfies the orthogonality condition for all subsequent times*

$$\left\langle \xi(\tau),\, \Sigma^{(m,n)}(a(\tau)) \right\rangle_{a(\tau)} = 0, \quad \tau \geq 0,\ (m, n) = (0, 0), (0, 1), (1, 1),$$

if and only if $\sigma = (g, z, a)$ *satisfies the modulation equations:*

$$\partial_\tau \sigma = \vec{F}(\sigma)\xi + \vec{M}(\sigma, \xi),$$

for some explicit vector fields $\vec{F}$, $\vec{M}$.

Proof.

Differentiating both sides of the equation

$$\left\langle \xi(\tau),\, \Sigma^{(m,n)}(a(\tau)) \right\rangle_{a(\tau)} = 0 \tag{7}$$

w.r.t. τ, we find that (7) holds for all $\tau \geq 0$ if and only if

1. (7) holds for $\tau = 0$;
2. For $\tau > 0$, there holds

$$\left\langle \dot{\xi}(\tau),\, \Sigma^{(m,n)}(a(\tau)) \right\rangle_{a(\tau)} = -\left\langle \xi(\tau),\, \partial_\tau \Sigma^{(m,n)}(a(\tau)) \right\rangle_{a(\tau)}. \tag{8}$$

The first point is in the assumption. Plugging the equation for $\dot{\xi}$ (which involves $\partial_\sigma W(\sigma)\dot{\sigma}$) into (8) , we get the equation for $\dot{\sigma}$. □

Linear correction

The linear subspace $\mathcal{S}$ in the Main Theorem removes all the zero-unstable modes of $L(a_0)$ from the configuration space $X^s(a_0)$. Hence, the codimension of $\mathcal{S}$ is the sum of the multipliciteis of all the non-positive eigenvalues of $L(a_0)$, which equals to

$$\operatorname{codim} \mathcal{S} = n + 2 + \frac{(n-k)(n-k+3)}{2}.$$

Hence, by the Modulation Lemma, if $\eta_0 \in \mathcal{S}$ and σ solves the modulation equation, then the flow $\xi(t)$ generated by η_0 remains orthogonal to $\Sigma^{(m,n)}(a)$, $(m,n) = (0,0),(0,1),(1,1)$ for all $\tau \geq 0$.

Need of quadratic correction

The twist here is that not all of the zero-unstable modes of the linearized operator $L(a)$ can be eliminated by the modulation method. Two classes of zero-unstable modes persist:

$$\Sigma^{(1,0)(i,0,0)}(a) := \frac{1}{\|y^i\|^2_{0,a}} y^i \quad (EV = -a),$$

$$\Sigma^{(2,0)(i,j,0)}(a) := \frac{1}{\|ay^i y^j - \delta_{ij}\|^2_{0,a}} (ay^i y^j - \delta_{ij}) \quad (EV = 0).$$

To eliminate these, we introduce the correction map Φ. This is quadratic in the sense that $\|\Phi(\eta)\|_{X^s} \lesssim \|\eta\|^2_{X^s}$ for $\eta \in \mathcal{S} \cap \mathcal{B}_\delta$. As we will see, $\Phi(\eta)$ incorporates certain unstable modes of $L(a_0)$.

Preliminaries for defining Φ

In order to define Φ, take a fixed path

$$(\sigma^{(0)}, \xi^{(0)}) \in Lip(\mathbb{R}_{\geq 0}, \Sigma) \times (C(\mathbb{R}_{\geq 0}, X^s) \cap C^1(\mathbb{R}_{\geq 0}, X^{s-2})).$$

Consider the following system, obtained by freezing coefficients in the graphical RMCF and the modulation equations at $(\sigma^{(0)}, \xi^{(0)})$:

$$\dot{\xi} = -L(a^{(0)})\xi - N(a^{(0)}, \xi^{(0)}) - \partial_\sigma W(\sigma^{(0)})\dot{\sigma}, \tag{LEξ}$$

$$\dot{\sigma} = \vec{F}(\sigma^{(0)})\xi + \vec{M}(\sigma^{(0)}, \xi^{(0)}), \tag{LEσ}$$

To (LEξ)-(LEσ) we associate the initial configurations

$$\sigma(0) = (\mathbf{1}, 0, a_0) \quad \text{for some fixed } a_0 > 1/2, \tag{ICσ}$$

$$\xi(0) = \eta_0 + \beta_i \Sigma^{(1,0)(i,0,0)}(a_0) + \gamma_{ij} \Sigma^{(2,0),(i,j,0)}(a_0). \tag{ICξ}$$

Here $\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S} \subset X^s(a_0)$ is fixed, and $\beta_i, \gamma_{ij} \in \mathbb{R}$ are to be chosen later as functions of η_0. Recall that

$$\Sigma^{(1,0)(i,0,0)}(a) := \frac{1}{\|y^i\|^2_{0,a}} y^i \quad (EV = -a), \tag{9}$$

$$\Sigma^{(2,0)(i,j,0)}(a) := \frac{1}{\|a y^i y^j - \delta_{ij}\|^2_{0,a}} (a y^i y^j - \delta_{ij}) \quad (EV = 0). \tag{10}$$

are the zero-unstable modes that persist modulation.

The solution map Ψ

Consider the solution map

$$\Psi : (\sigma^{(0)}, \xi^{(0)}) \mapsto \text{ the unique solution } (\sigma, \xi) \text{ to (LE}\xi\text{)-(IC}\xi\text{)}. \quad (11)$$

- Using standard parabolic theory, one can show that this map is well-defined.
- Hereafter we want to show that Ψ is a contraction in a suitable space of dissipating paths.
- Then the fixed point of Ψ will be a dissipating solution to the graphical RMCF.

Definition

Fix $0 < \delta \ll 1$. The space $\mathcal{A}_\delta = \mathcal{A}_\delta^\sigma \times \mathcal{A}_\delta^\xi$ consists of

$$(\sigma, \xi) \in Lip(\mathbb{R}_{\geq 0}, \Sigma) \times (C(\mathbb{R}_{\geq 0}, X^s) \cap C^1(\mathbb{R}_{\geq 0}, X^{s-2})),$$

s.th. the following holds:

1. $\sigma(0)$ is as in (ICσ), with $a_0 \geq \frac{1}{2} + 2\delta$;
2. For some fixed $c_0 > 0$, there hold the decay estimates

$$|\dot{\sigma}(\tau)| \leq c_0 \delta \langle \tau \rangle^{-2}, \quad \tau \geq 0, \tag{12}$$

$$\|\xi(\tau)\|_s \leq \delta \langle \tau \rangle^{-2}, \quad \tau \geq 0; \tag{13}$$

3. Certain pivot condition (technical but easy to verify).

The contraction scheme

We want to show

1. $\Psi(\mathcal{A}_\delta) \subset \mathcal{A}_\delta$;
2. $\Psi : \mathcal{A}_\delta \to \mathcal{A}_\delta$ is a contraction w.r.t. a suitable norm on $\mathcal{A}_\delta$.

Remark. Consider the initial configuration (ICξ).

The constants β_i and γ_{ij} are to be determined later as a function of $\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S}$. Hence, the map Ψ depends only on η_0.

Indeed, if $\eta_0 = 0$, then it is easy to see that the fixed point of $\Psi(\cdot, 0)$ is just the vector $(\sigma, \xi) \equiv (\sigma(0), 0)$ in $\mathcal{A}_\delta$. This corresponds to the trivial static solution (=cylinder of radius $\sqrt{k/a_0}$).

Definition (the quadratic correction Φ)

For $\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S}$, define

$$\Phi(\eta_0) := \beta_i(\eta_0)\Sigma^{(1,0)(i,0,0)}(a_0) + \gamma_{ij}(\eta_0)\Sigma^{(2,0),(i,j,0)}(a_0), \qquad (14)$$

where β_i, γ_{ij} are some functions of η_0, s.th. $\Psi = \Psi(\cdot, \eta_0)$ has a unique fixed point in $\mathcal{A}_\delta$ (c.f. the initial condition (ICξ) for ξ).

Proof of the Main Theorem.

By construction of Ψ, if it has fixed point in $\mathcal{A}_\delta$, then this fixed point is a global solution to the graphical RMCF dissipating to 0 in X^s-norm as $\tau \to \infty$. □

The heart of the matter is to show $\Psi(\mathcal{A}_\delta) \subset \mathcal{A}_\delta$. Given η_0, we construct explicit numbers $\beta_i(\eta_0)$, $\gamma_{ij}(\eta_0)$ in the initial condition (ICξ) that fulfills this mapping property.

Theorem (Choice of β, γ)

For every $\eta_0 \in \mathcal{B}_\delta \cap \mathcal{S}$ and every fixed path $(\sigma^{(0)}, \xi^{(0)}) \in \mathcal{A}_\delta$, there exist unique coefficients β_i, γ_{ij}, depending on the choice of $\sigma^{(0)}$, $\xi^{(0)}$ only, s.th. the solution to (LEξ)-(ICξ) *lies in $\mathcal{A}_\delta$. Moreover, there hold the quadratic estimates*

$$\left|\beta_i(\sigma^{(0)}, \xi^{(0)})\right| \lesssim \delta^2, \tag{15}$$

$$\left|\gamma_{ij}(\sigma^{(0)}, \xi^{(0)})\right| \lesssim \delta^2. \tag{16}$$

Lemma (mechanism for choosing β and γ)

Fix two functions $a(t) \geq 0$ *and* $f \in L^1(\mathbb{R}_{\geq 0}, \mathbb{R})$*.Consider the Cauchy problem for* $x : \mathbb{R}_{\geq 0} \to \mathbb{R}$*:*

$$\dot{x} - a(t)x = f, \tag{17}$$

$$x(0) = x_0 \in \mathbb{R}. \tag{18}$$

There exists a unique solution with $\lim_{t\to\infty} x(t) = 0$ *if and only if*

$$x_0 = -\int_0^\infty f_i(t')e^{-\int_0^{t'} a}\,dt' \text{ in (18).} \tag{19}$$

Moreover, if (19) *holds, then the solution to* (17)-(18) *is given by*

$$x(t) = -\int_t^\infty f(t')e^{-\int_t^{t'} a},\, d\tau'. \tag{20}$$

Sketch Proof of the Choice Theorem

1. Let $(\sigma, \xi) := \Psi(\sigma^{(0)}, \xi^{(0)})$. By assumption, $(\sigma^{(0)}, \xi^{(0)})$ satisfies the decay estimate

$$\left|\dot{\sigma}^{(0)}(\tau)\right| \leq c_0 \delta \left\langle \tau \right\rangle^{-2}, \quad \tau \geq 0,$$
$$\left\|\xi^{(0)}(\tau)\right\|_s \leq \delta \left\langle \tau \right\rangle^{-2}, \quad \tau \geq 0.$$

We want to show the same for (σ, ξ).The key is to check the decay conditions for ξ. Then the decay of $\dot{\sigma}$ follows from the modulation equation.

2. Let $P^{(m,n)}(\tau)$ be the projection onto the span of the zero-unstable modes span $\left\{\Sigma^{(m,n)(i,j,l)}(a^{(0)}(\tau))\right\}$ of $L(a^{(0)}(\tau))$, and write $\xi^{(m,n)} := P^{(m,n)}\xi$. Let $Q := 1 - \sum P^{(m,n)}$ and write $\xi_S = Q\xi = \xi - \sum \xi^{(m,n)}$ (=stable projection w.r.t. $L(a^{(0)})$).
Due to the modulation equation, we know

$$\xi^{(m,n)} = 0 \text{ for } (m,n) = (0,0), (0,1), (1,1).$$

Now we expand

$$\xi = \xi_S + \sum_{m=1,2} \xi^{(m,0)},$$

and plug this expansion into the graphical RMCF for ξ:

$$\dot{\xi} = -L(a^{(0)})\xi - N(a^{(0)}, \xi^{(0)}) - \partial_\sigma W(\sigma^{(0)})\dot{\sigma}.$$

Using the orthogonality among various eigenfunctions of the self-adjoint operator $L(a^{(0)}(\tau))$ in the space $X^s(a^{(0)}(\tau))$, $s \geq 2$, we get the following system:

$$\dot{\xi}_S - QL(a^{(0)})\xi_S = -QN(a^{(0)}, \xi^{(0)}), \tag{21}$$

$$\dot{\beta}_i - a^{(0)}\beta_i = f_i, \tag{22}$$

$$\dot{\gamma}_{ij} = h_{ij}, \tag{23}$$

where

$$f_i = -\left\langle N(a^{(0)}, \xi^{(0)}),\ \Sigma^{(1,0)(i,0,0)}(a^{(0)})\right\rangle_{a^{(0)}}, \tag{24}$$

$$h_{ij} = -\left\langle N(a^{(0)}, \xi^{(0)}),\ \Sigma^{(2,0)(i,j,0)}(a^{(0)})\right\rangle_{a^{(0)}}. \tag{25}$$

The initial configurations associated to (21)-(23) are resp. the three terms in the initial condition for ξ:

$$\xi(0) = \eta_0 + \beta_i \Sigma^{(1,0)(i,0,0)}(a_0) + \gamma_{ij}\Sigma^{(2,0),(i,j,0)}(a_0) \quad (\mathrm{IC}\xi)$$

Since (21)-(23) are already decoupled, in what follows we consider the Cauchy problem for ξ_S, β_i, γ_{ij} separately.
First, for the equation (21) for the stable part ξ_S, we can use standard propagator estimate to show that

$$\|\xi_S(\tau)\|_s \le \delta e^{-c\tau}, \quad \tau \ge 0, \tag{26}$$

where c is an absolute constant.

Next, applying the ODE Lemma to the equation for β

$$\dot{\beta}_i - a^{(0)}\beta_i = f_i,$$

we find that $\lim_{\tau\to\infty}\beta_i(\tau) = 0$ if and only if the initial configuration is given by

$$\beta_i(0) = -\int_0^\infty f_i(\tau')\lambda(\tau')^{-1}\,d\tau' \quad \left(\lambda(\tau) = e^{\int_0^\tau a(\tau')\,d\tau'}\right). \quad (27)$$

This integral indeed converges, since $\lambda(\tau) \geq 1$, and some nonlinear estimate that shows $|f_i(\tau)| \lesssim \langle\tau\rangle^{-4}$.

Choose $\beta_i(0)$ as in (27). Then $\beta_i(\tau)$ is given by the ODE Lemma:

$$\beta_i(\tau) = -\int_{\tau}^{\infty} f(\tau')\frac{\lambda(\tau)}{\lambda(\tau')}\, d\tau'. \tag{28}$$

For all $\tau \geq 0$, this function satisfies

$$\begin{aligned} |\beta_i(\tau)| &\leq \int_{\tau}^{\infty} \left|f_i(\tau')\right| e^{-\int_{\tau}^{\tau'} a}\, d\tau' \\ &\leq \int_{\tau}^{\infty} \left|f_i(\tau')\right| d\tau' \\ &\leq C\delta^2 \left\langle \tau \right\rangle^{-3}. \end{aligned} \tag{29}$$

Here the last inequality follows from $|f_i| \lesssim \delta^2 \left\langle \tau \right\rangle^{-4}$, due to some nonlinear estimate as before.

We conclude from (29) that

$$|\beta_i(0)| \lesssim \delta^2. \tag{30}$$

This gives quadratic estimate (15).

By (29) and some technical interpolation inequalities, we conclude

$$\left\| P^{(1,0)} \xi(\tau) \right\|_s \leq \delta \left\langle \tau \right\rangle^{-2}, \tag{31}$$

provided δ is sufficiently small.Using the exact same argument we can determine the unique numbers $\gamma_{ij}(\eta) = O(\delta^2)$ that make

$$\left\| P^{(2,0)} \xi(\tau) \right\|_s \leq \delta \left\langle \tau \right\rangle^{-2}. \tag{32}$$

This proves the Choice Theorem. □

Thanks for your attention