Ginzburg-Landau equations on hyperbolic surfaces

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Overview

- Based on joint work arXiv:2203.14179 with NM Ercolani (Tuscon) and IM Sigal (Toronto) during Spring research visit.
- Main result: existence theory for Ginzburg-Landau (GL) equations on non-compact Riemann surfaces with constant negative curvature ($\equiv$ hyperbolic surfaces).
- Techniques: Lyapunov-Schmidt reduction and bifurcation analysis. No variational/Bogolmonyi structure is used.

Setup

We consider the Ginzburg-Landau equations on a line bundle E over a Riemann surface (Σ, h):

$$\begin{aligned} -\Delta_a \psi &= \kappa^2 \left(1 - |\psi|^2\right) \psi, \\ d^* da &= \operatorname{Im}\left(\bar{\psi} \nabla_a \psi\right). \end{aligned} \tag{GL}$$

- $\kappa > 0$ is a fixed (dimensionless) material parameter.
- $(\psi, a) =$ (section,1-form)-pair on the line bundle E.
- ∇_a is the covariant derivative induced by a.
- $-\Delta_a = \nabla_a^* \nabla_a$ (Note that ∇_a^* depends on the metric h).
- d denotes the exterior derivative on Σ.

Geometric setting

Surface Σ: By the Uniformization theorem, every hyperbolic surface Σ is of the form

$$\Sigma \cong \mathbb{H}/\Gamma.$$

Here $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and Γ is a Fuchsian group (i.e. discrete subgp. of $SL(2,\mathbb{R})$), acting on $\mathbb{H}$ by Möbius transform:

$$\gamma z = \frac{az+b}{cz+d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hyperbolic metric on Σ: For each $r > 0$, let

$$h_r = \frac{r}{(\operatorname{Im} z)^2} dz \otimes d\bar{z} \implies (\Sigma, h_r) \text{ has const. curvature } -1/r.$$

Assumption on $\Sigma \cong \mathbb{H}/\Gamma$

Our existence theory holds on any hyperbolic surface Σ with finite area, finitely many cusps, and no elliptic points (which are conditions on Γ).

Example

There exists an infinite family of distinct Σ's with above properties: the arithmetic surfaces $\Sigma \cong \mathbb{H}/\Gamma(N)$, $N \geq 2$, where

$$\Gamma(N) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : a \equiv d \equiv 1, b \equiv c \equiv 0 \mod N \right\}.$$

Fundamental domain

A fundamental domain $F_\Sigma \subset \mathbb{H}$ of $\Sigma = \mathbb{H}/\Gamma$ ($\Gamma =$ a Fuchsian group) is a connected open subset such that no two points of F_Σ are equivalent under Γ and $\mathbb{H} = \bigcup_\gamma \gamma \bar{F}_\Sigma$ ($\bar{F}_\Sigma \equiv$ closure of F_Σ).

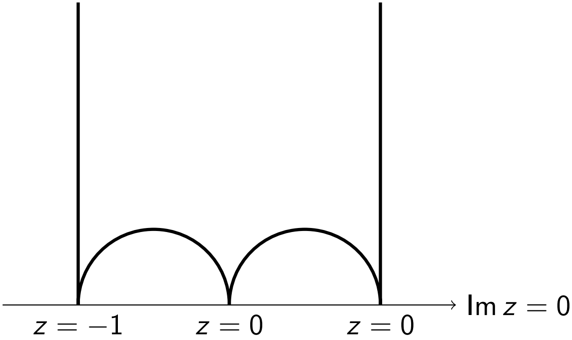

Figure: A fundamental domain of $\Gamma(2)$ in $\mathbb{H}$ with three cusps.

Configuration space X^k for (GL)

The point of all previous discussions is to show that existence theory for (GL) on line bundle $E \to \Sigma$ is equivalent to solving (GL) in $X^k \equiv X^k_{\Sigma,E}$, the Sobolev space of order k of (function, vector field)-pairs on F_Σ with gauge-periodic boundary conditions:

$$\gamma^*\Psi(z) = \rho(\gamma, z)\Psi(z),$$
$$\gamma^*A(z) = A(z) + i\rho(\gamma, z)^{-1} d\rho(\gamma, z),$$

for every $z \in \partial F_\Sigma$, $\gamma \in \Gamma$ and some $\rho(\gamma, z) : \Gamma \times \mathbb{H} \to U(1)$ with

$$\rho(\gamma\gamma', z) = \rho(\gamma, \gamma' z)\rho(\gamma', z) \quad \left(\gamma, \gamma' \in \Gamma,\, z \in \mathbb{H}\right).$$

The choice of ρ is determined by the topology of E.

Constant curvature solutions to GL

On $E \to (\Sigma, h_r)$, (GL) has the following const. curvature solutions:

$$\psi \equiv 0, \quad a = a^b,$$

where ψ is the zero-section on the line bundle E, and a^b is a constant curvature connection satisfying

$$da^b = b\omega_r \text{ with } b = b(\Sigma, E, r) := \frac{2\pi \deg E}{|\Sigma|_r}. \tag{b}$$

The value of b in (b) is determined by the Chern-Weil relation:

$$\frac{1}{2\pi}\int_\Sigma da = \deg E \quad \forall \text{gauge-periodic 1-form } a \text{ with } \left|\int_\Sigma da\right| < \infty.$$

Auxiliary functions

In what follows, we fix line bundle $E \to \Sigma$, and vary the metric $h = h_r$. So the only free parameter is $r > 0$ ($\equiv$curvature on Σ).

To state our main result, we define the Abrikosov function, $\beta = \beta(b(r))$, as

$$\beta(r) := \min \left\{ \|\xi\|_{L^4}^4 : \xi \in \mathrm{Null}(-\Delta_{a^b} - b), \|\xi\|_{L^2} = 1 \right\},$$

and the threshold Ginzburg-Landau parameter, $\kappa_c = \kappa_c(\beta(r))$, as

$$\kappa_c(r) := \sqrt{\frac{1}{2}\left(1 - \frac{1}{\beta(r)}\right)}.$$

Theorem (existence theory for GL on $E \to (\Sigma, h_r)$)

Let $b_0 := 2\pi \deg E / |\Sigma|$. *There exists a family of solutions*

$$(\psi_{s(r)}, a_{s(r)}), \tag{1}$$

*to (*GL*), each sitting in a nbhd.* $U \subset X^k$ *around the const. curvature solution* $(0, a^{b_0/r})$*, labeled by parameter* $r > 0$ *with*

$$0 < \left|\kappa^2 r - b_0/r\right| \ll 1, \quad (\kappa - \sqrt{b_0/r})(\kappa - \kappa_c(r)) > 0, \tag{C}$$

and

$$s = s(r) \in \mathbb{R}^D, \quad D := \dim_{\mathbb{C}} \mathrm{Null}(-\Delta_{a^b} - b_0/r)$$

is an analytic curve, and satisfies $0 < |s| \ll 1$.

Remarks on the main theorem

- The existence condition (C) was first isolated as a criterion for the existence of the Abrikosov vortex lattice in a series of works of Sigal-Tzaneteas' (e.g. Contemp. Math. 2011).
- Condition (C) gives rise to two critical magnetic fields: $b = \kappa\sqrt{b_0}$ from the first part, and $b = \kappa^2$ from the second.
- Physically, the number $\kappa^2 r$ corresponds to the average magnetic field in superconductors. Hence, condition (C) can be compared to S. Bradlow's existence condition for magnetic vortices on compact Riemann surfaces.

Idea of the proof of the main theorem

Step 0: rescaling

Under a suitable rescaling, (GL) on $E \to (\Sigma, h_r)$ (recall $-1/r < 0$ is the curvature on Σ) is equivalent to the rescaled GL equations,

$$\begin{aligned} -\Delta_a \psi &= \kappa^2 \left(r - |\psi|^2 \right) \psi, \\ d^* da &= \operatorname{Im} \bar{\psi} \nabla_a \psi, \end{aligned} \tag{RGL}$$

posed on the Sobolev space X^k, $k \geq 2$ over (F_Σ, h_1).

Then the proof of the main theorem consists of two-part analysis of (RGL). First, we linearize (RGL) around the constant curvature solution $(\psi, a) = (0, a^b)$.

Step 1: Linear analysis

The linearized problem associated to (RGL) reduces to understanding the spectral properties of the Laplacian $-\Delta_{a^b}$ associated to a constant curvature connection a^b, acting on L^2-sections of the unitary line bundle $E \to (\Sigma, h_1)$.

Theorem (spectrum of $-\Delta_{a^b}$)

1. $-\Delta_{a^b} \geq b$.
2. *Denote by $\mathcal{S}(\Sigma) \equiv \mathcal{S}_{2b}(\Sigma)$ the space of cusp forms on Σ with weight $2b$ (i.e. L^2-automorphic functions with weight $2b$). Then b is an eigenvalue of $-\Delta_{a^b} \iff \mathcal{S}(\Sigma) \neq \emptyset$.*
3. $\sigma_{\mathrm{ess}}(-\Delta_{a^b}) = [\frac{1}{4} + b^2, \infty)$ *(note that $\frac{1}{4} + b^2 \geq b$).*

Step 2: nonlinear analysis

- Next, we use Lyapunov-Schmidt reduction to show that a non-trivial branch of solution of the form bifurcates from the constant curvature solutions, provided the metric on Σ satisfies the main existence condition (C), and a key bifurcation equation is satisfied.
- This bifurcation equation amounts to a finite system of algebraic equations, which we solve directly using Weierstrass Preparation Theorem and Hartogs' extension theorem.

Thank you for your attention