Quasi-locality and recursive monotonicity estimates for quantum evolutions

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Overview

- Based on joint works with IM Sigal (Toronto).
- Main result: quasi-locality theorems for a large class of time-dependent Schrödinger equations (nonlocal, non-autonomous, one-/many-body).
- Techniques: finite iterations of recursive monotonicity estimates for time-dependent observables that identify probability leakage.

Setup

Consider the following Schrödinger equation:

$$i\partial_t \psi = H(t)\psi. \tag{SE}$$

- State: $\psi(\cdot, t) \in \mathfrak{h} := L^2(\mathbb{R}^d, \mathbb{C})$, $d \geq 1$, $t \in \mathbb{R}$.
- Hamiltonian: $H(t) = H_0 + V(t)$ where

$$H_0[\psi](x) = \int_{y \in \mathbb{R}^d} (\psi(x) - \psi(y)) K(x, y),$$

$$K(y, x) = \overline{K(x, y)} \quad \forall x, y \in \mathbb{R}^d,$$

$$V(t) \in \mathcal{B}(\mathfrak{h}) \quad \forall t \in \mathbb{R}.$$

Dynamics

Standing assumption. H_0 is self-adjoint on a dense domain $\mathcal{D} \equiv \mathcal{D}(H_0) \subset \mathfrak{h}$.

$\implies$ $H(t) = H_0 + V(t)$ is self-adjoint on $\mathcal{D}$ and has bounded propagator $U(t,s)$ w/ $U(t,s)\psi_s = \psi_t$.

$\implies$ Evolution of observables, A, dual to the evolution of states $\psi \mapsto U_t\psi$ (where $U_t \equiv U(t,0)$) w.r.t. the coupling $(A,u) \mapsto \langle \psi,\, A\psi \rangle$, is given by

$$\alpha_t(A) := U(t,0)^* A U(t,0). \quad \text{(Heisenberg picture)}$$

Main assumptions

Recall $H(t) = H_0 + V(t)$ with nonlocal operator

$$H_0[\psi](x) = \int_{y \in \mathbb{R}^d} (\psi(x) - \psi(y)) K(x, y). \qquad (H_0)$$

Main technical assumption: $\exists n \geq 1$ s.th. the $(n+1)$-th moments of K are all finite, i.e.,

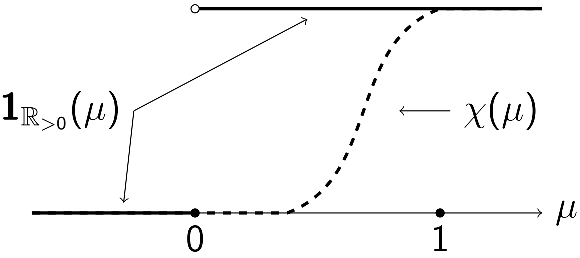
$$\sup_{\substack{1 \leq p \leq n+1 \\ x \in \mathbb{R}^d}} \int_{y \in \mathbb{R}^d} |K(x, y)| \, |x - y|^p \leq \kappa < \infty. \qquad (\kappa)$$

Propagation identifier observables

Let $c > \kappa$, $\phi \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{R})$. For $s > 0$, define

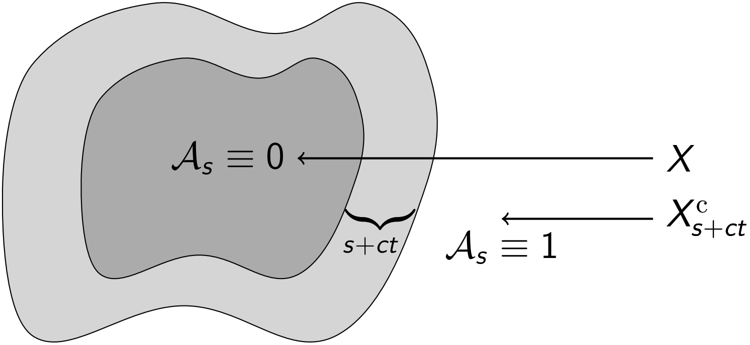
$$\begin{array}{rcccl} \mathcal{A}_s & : & \mathbb{R} \times \mathcal{X} & \longrightarrow & C(\mathbb{R}^d, \mathbb{R}) \\ & & (t, \chi) & \longmapsto & \chi\left(\frac{\phi - ct}{s}\right) \end{array}, \qquad \text{(PIO)}$$

where $\mathcal{X} \subset C^\infty(\mathbb{R})$ consists of suitable cut-off functions χ with compactly supported derivatives:



Remarks on the PIOs

The PIOs $\mathcal{A}_s(t,\chi)$ play a central role in our analysis.
Let $\phi(x) := \mathrm{dist}(\{x\}, X) \equiv d_X$ in (PIO) for $X \subset \mathbb{R}^d$, so that $\mathcal{A}_s(t,\chi) \equiv \chi\left(\frac{d_X - ct}{s}\right)$ is localized outside the light cone $X_{ct} := \{d_X(x) \leq ct\}$:



Monotonicity estimates for PIO

Since $\mathcal{A}_s(t,\chi)$ identifies the probability leakage ouside the light cone, decay estimates on $\mathcal{A}_s(t,\chi)$ amount to control on the probability tails.

Theorem (Monotonicity estimate)

Suppose (κ) *holds for* $n \geq 1$. *Then,* $\forall c > \kappa, \chi \in \mathcal{X}$, $\exists\ C > 0, \xi \in \mathcal{X}$ *s.th.* $\forall s > 0,\ t \geq 0,\ \psi_t = U_t\psi \in \mathcal{D}$,

$$\langle \psi_t,\, \mathcal{A}_s(t,\chi)\psi_t\rangle \leq \langle \psi,\, \mathcal{A}_s(0,\chi)\psi\rangle + C\left(s^{-1}\langle \psi,\, \mathcal{A}_s(0,\xi)\psi\rangle + ts^{-(n+1)}\|\psi\|^2\right). \quad \text{(ME)}$$

Remarks on the ME

(ME) is derived from bootstrapping a type of recursive differential inequalities, recently obtained e.g. [J. Faupin, M. Lemm, I.M. Sigal, PRL 128 (2022), 150602.]: $\forall \chi \in \mathcal{X}$, $\exists\ \delta$, $C > 0$, $\xi \in \mathcal{X}$ s.th.

$$\partial_t \alpha_t\left(\mathcal{A}_s(t,\chi)\right) \leq -\,\delta s^{-1} \alpha_t\left(\mathcal{A}_s(t,\chi')\right) + C\left(s^{-2}\alpha_t\left(\mathcal{A}_s\left(t,\xi'\right)\right) + s^{-(n+1)}\right).$$

This differential ineq. is 'recursive monotone' because the second, remainder term on the r.h.s. is of the same form as the leading, negative term.

Quasi-locality of (SE): Heuristics

Suppose for now elements in $\mathcal{X}$ are sharp cut-off functions supported in $(0, \infty)$. Then, for $s > t \geq 0$, (ME) implies the following decay estimate for the states $\psi_t = U_t \psi$ evolving according to (SE):

$$\left\| \chi_{\{\phi \geq ct\}} \psi_t \right\|^2 \leq (1 + Ct^{-1}) \left\| \chi_{\{\phi \geq 0\}} \psi \right\|^2 + C \left\| \psi \right\|^2 t^{-n}.$$

If, moreover, the initial state ψ is supported outside the set $\{\phi \geq 0\}$, then the leading term above vanishes and we obtain an $O(t^{-n/2})$ estimate on the probability tail outside the light cone $\{\phi \leq ct\}$.

Quasi-locality of (SE)

With the choice $\phi(x) = \mathrm{dist}(\{x\}, X)$ for $X \subset \mathbb{R}^d$ in (PIO), we can show, by (ME) and some geometric inequalities for $\mathcal{A}_s(t, \chi) \equiv \chi(s^{-1}(\phi - ct))$, that:

Theorem (Localization of states)

Suppose (κ) *holds for* $n \geq 1$. *Then* $\forall\ c > \kappa$, $\exists C = C(n, c, \kappa) > 0$ *s.th.* $\forall X \subset \mathbb{R}^d$, $\psi \in \mathcal{D}$ *with* $\chi_{X^c}\psi = 0$, *and* $\psi_t = U_t\psi$, $t \geq 0$,

$$\left\langle \psi_t, \, \chi_{\{d_X(x) \geq ct\}} \psi_t \right\rangle \leq C t^{-n} \|\psi\|^2 . \quad \text{(Quasi-locality)}$$

(L.h.s.= probability leakage outside the line cone.)

Conclusion

- Pseudolocality theorems and monotonicity estimates are powerful tools in the study of geometric flows, as they impose general constraints on evolving geometric objects that are otherwise hard to keep track of.
- We have proved parallel results for nonlocal non-autonomous quantum evolutions.
- Our method is robust and persists the second-quantization (i.e. results are valid for general quantum many-body systems).